GEOMETRICAL EXERCISES

WENTWORTH

GEOMETRICAL EXERCISES.

EXERCISES

CONTAINED IN

WENTWORTH'S GEOMETRY,

WITH KEY,

FOLLOWED BY A SELECTION OF

MISCELLANEOUS EXERCISES

FOR PRACTICE.

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PREFACE.

THIS volume has been prepared at the solicitation of many teachers. The solutions of the exercises contained in Wentworth's Geometry are those given by the students of the Academy, and are believed to be especially valuable as showing the methods which naturally suggest themselves to the minds of beginners.

The miscellaneous exercises are of various degrees of difficulty, and from them teachers can select such as are best adapted to their pupils.

This book is intended exclusively for teachers.

PHILLIPS EXETER ACADEMY,
November, 1879.

GEOMETRICAL EXERCISES.

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Ex. 1. If an angle be a right angle, what is its complement? *Ans.* 0.

Ex. 2. If an angle be a right angle, what is its supplement? *Ans.* 1 rt. $\angle$.

Ex. 3. If an angle be three-fifths of a right angle, what is its complement? *Ans.* $\frac{2}{5}$ rt. $\angle$.

Ex. 4. If an angle be three-fifths of a right angle, what is its supplement? *Ans.* $\frac{7}{5}$ rt. $\angle$s.

Ex. 5. Show that the bisectors of two vertical angles form one and the same straight line.

Let CFA and BFD be two vertical $\angle$s; and let FE and FH be their respective bisectors. $\angle\, CFA = \angle\, BFD$, $\therefore \angle\, EFA = \angle\, BFH$ (Ax. 7). Now $\angle$s $EFA + CFE + CFB = 2$ rt. $\angle$s. $\therefore \angle$s $ECF + CFB + BFH = 2$ rt. $\angle$s; or, $\angle$s $EFB + BFH = 2$ rt. $\angle$s. $\therefore FE$ and FH form the same straight line (§ 51).

Ex. 6. Show that the two straight lines which bisect the two pairs of vertical angles are perpendicular to each other.

Let COB and AOD, COA and BOD be the two pairs of vertical $\angle s$; and EF and HM their respective bisectors. Then $\angle HOC = \angle MOB$ (being halves of equal $\angle s$); and $\angle COE = \angle BOE$ (being halves of the same $\angle$). $\therefore \angle HOE = \angle MOE$, $\therefore EF$ is $\perp$ to HM (§ 26).

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Ex. 1. Show that the sum of the distances of any point in a triangle from the three angles of the triangle is greater than half the sum of the sides of the triangle.

Let ABC be the $\triangle$, and O any point within it. Draw OA, OB, OC. Now $OA + OB > AB$; $OA + OC > AC$; $OB + OC > BC$. $\therefore 2\,OA + 2\,OB + 2\,OC > AB + AC + BC$. $\therefore OA + OB + OC > \dfrac{AB + AC + BC}{2}$.

Ex. 2. Show that the locus of all the points at a given distance from a given straight line AB consists of two parallel lines, drawn on opposite sides of AB, and at the given distance from it.

Draw CM and DH on opposite sides of AB, and at the given distance from it. Then CM and

DH are the required locus, and since they are everywhere equally distant from *AB*, they are ∥ to it, and hence ∥ to each other (§ 75).

Ex. 3. Show that the two equal straight lines drawn from a point to a straight line make equal acute angles with that line.

Let *CD* and *CE* be two equal straight lines drawn from the point *C*, and meeting the straight line *AB* at *D* and *E*. Draw *CF* ⊥ to *AB*. Now $FD = FE$ (§ 57). Fold over *CFD* on *CF* as an axis; then △ *CFD* will coincide with △ *CFE*. Hence $\angle\ CDF = \angle\ CEF$.

Ex. 4. Show that, if two angles have their sides perpendicular, each to each, they are either equal or supplementary.

Let *CHD* and *AOB* be two ∠s having *HC* ⊥ to *OB* and *HD* ⊥ to *OA*. Revolve ∠ *CHD* about the vertex *H* as a pivot until each of its sides describes a rt. ∠. The sides of the two ∠s will then be ∥, two and two. If the sides lie in the same direction, or opposite directions, from their vertices, the ∠s are equal (§ 77); if two of the sides lie in the same direction from their vertices, while the other two sides lie in opposite directions, the two ∠s are supplements of each other (§ 78).

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Ex. If the equal sides of an isosceles triangle be produced, show that the angles formed with the base by the sides produced are equal.

Let CBD and BCE be the two $\angle$s formed by producing the equal sides AB and AC of the isosceles $\triangle$ ABC. Since $\angle$s ABC and ACB are equal (§ 112), their supplements CBD and BCE are equal.

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Ex. Show that an equiangular triangle is also equilateral.

Let ABC be an equiangular $\triangle$. Since $\angle B = \angle C$, side $AC =$ side AB (§ 114). Since $\angle A = \angle B$, side $BC =$ side AC. $\therefore \triangle ABC$ is equilateral.

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Ex. ABC and ABD are two triangles on the same base AB, and on the same side of it, the vertex of each triangle being without the other. If AC equal AD, show that BC cannot equal BD.

Let CAB and DAB be two $\triangle$s on the same base, AB, and on the same side of it, the vertex of each $\triangle$ being without the other, and let $AC = AD$. Join the vertices C and D. Since $AC = AD$, $\angle ACD = \angle ADC$ (112). But $\angle BCD <$ $\angle ACD$ and $\angle BDC > \angle ADC$. $\therefore \angle BDC$ $> \angle BCD$. $\therefore BC > BD$ (§ 117).

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Ex. If the angles ABC and ACB, at the base of an isosceles triangle, be bisected by the straight lines BD, CD, show that DBC will be an isosceles triangle.

$\angle ABC = \angle ACB$, $\therefore \angle CBD = \angle BCD$ (being halves of equal $\angle$s). $\therefore \triangle DBC$ is isosceles (§ 114).

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Ex. 1. Show that the interior angles of a hexagon are equal to eight right angles.

Since the interior $\angle$s of a polygon of n sides is equal to $2(n-2)$ rt. $\angle$s (§ 157), the interior $\angle$s of a hexagon equal $2(6-2)$ rt. $\angle$s $= 8$ rt. $\angle$s.

Ex. 2. Show that each angle of an equiangular pentagon is $\frac{6}{5}$ of a right angle.

Since each angle of an equiangular polygon of n sides is equal to $\frac{2(n-2)}{n}$ rt. $\angle$s (158), each angle of an equiangular pentagon $= \frac{2(5-2)}{5}$ rt. $\angle$s $= \frac{6}{5}$ rt. $\angle$s.

Ex. 3. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

Let $n =$ the number of sides. Then $\frac{2(n-2)}{n}$ rt. $\angle$s $=$ the value of each $\angle$ (§ 158). But each $\angle$ $= \frac{7}{4}$ rt. $\angle$s. $\therefore \frac{2(n-2)}{n} = \frac{7}{4}$. $\therefore n = 16$.

Ex. 4. How many sides has the polygon the sum of whose interior angles is equal to the sum of its exterior angles?

The exterior $\angle$s of a polygon $= 4$ rt. $\angle$s (§ 159); and the interior angles $= 2\ (n - 2)$ rt. $\angle$s (§ 157). $\therefore 2\ (n - 2) = 4. \quad \therefore n = 4.$

Ex. 5. How many sides has the polygon the sum of whose interior angles is double that of its exterior angles?

$2\ (n - 2) = 8. \quad \therefore n = 6.$

Ex. 6. How many sides has the polygon the sum of whose exterior angles is double that of its interior angles?

In this case $4\ (n - 2) = 4. \quad \therefore n = 3.$

Ex. 7. Every point in the bisector of an angle is equally distant from the sides of the angle; and every point not in the bisector, but within the angle, is unequally distant from the sides of the angle.

I. Let A be the given $\angle$, and AB its bisector. From any point in AB, as O, draw OC and OD $\perp$ to the sides of the $\angle$. In the rt. $\triangle$s AOC and AOD, $AO = AO$, $\angle\ OAC = \angle\ OAD$ (cons.). $\therefore \triangle\ AOC = \triangle\ AOD$ (§ 110). $\therefore OC = OD$.

II. Let E be any point within the $\angle$, but not in

the bisector AB. Draw EF and $ED \perp$ to the sides of the $\angle$, and let ED intersect the bisector AB at O. From O draw $OC \perp$ to the side AF, and join EC. Now $EF < EC$ (§ 52), and $EC < EO + OC$; that is, $EC < EO + OD$. $\therefore EC < ED$. Still more, then, is $EF < ED$.

Ex. 8. BAC is a triangle having the angle B double the angle A. If BD bisect the angle B, and meet AC in D, show that BD is equal to AD.

$\angle A = \frac{1}{2} \angle B$ (hyp.); and $\angle ABD = \frac{1}{2} \angle ABC$ (cons.). $\therefore \angle A = \angle ABD$. $\therefore BD = AD$ (§ 114).

Ex. 9. If a straight line drawn parallel to the base of a triangle bisect one of the sides, show that it bisects the other also; and that the portion of it intercepted between the two sides is equal to one-half the base.

Let ABC be a $\triangle$, and let DE be parallel to the base BC, and bisect the side AB at D. Draw $EF \parallel$ to AB. Then $EF = DB$ (§ 135). But $DB = AD$ (cons.). $\therefore EF = AD$. Now $\angle ADE = \angle B$ (ext.-int. $\angle$s), and $\angle B = \angle EFC$ (ext.-int. $\angle$s); $\therefore \angle ADE = \angle EFC$. Also, $\angle CEF = \angle A$ (ext.-int. $\angle$s). $\therefore \triangle ADE = \triangle EFC$ (§ 107). $\therefore AE = EC$, and $\therefore DE$ bisects AC. Also $FC = DE$. But $BF = DE$ (§ 135). $\therefore FC = BF$. $\therefore DE = \frac{1}{2} BC$.

Ex. 10. *ABCD* is a parallelogram, *E* and *F* the middle points of *AD* and *BC* respectively; show that *BE* and *DF* will trisect the diagonal *AC*.

Let *BE* and *DF* intersect the diagonal *AC* at the points *H* and *K*. The lines *ED* and *BF* are equal and $\parallel$ (being halves of the opposite sides of a $\square$). $\therefore$ *BE* is $\parallel$ to *DF* (§ 136). In the $\triangle$ *ADK*, *EH* is $\parallel$ to the base *DK*, and bisects the side *AD*. $\therefore$ *EH* bisects *AK* at *H* (Ex. 9). $\therefore AH = HK$. In the $\triangle$ *CBH*, *FK* is $\parallel$ to the base *BH* and bisects the side *CB*. $\therefore$ *FK* bisects the side *CH* at *K*. $\therefore CK = KH$. $\therefore$ *AH*, *HK*, and *KC* are equal. That is, *AC* is trisected.

Ex. 11. If from any point in the base of an isosceles triangle parallels to the equal sides be drawn, show that a parallelogram is formed whose perimeter is equal to the sum of the equal sides of the triangle.

Let *ABC* be an isosceles $\triangle$, and *D* any point in the base *BC*. Draw *DE* $\parallel$ to *CA*, and *DF* $\parallel$ to *BA*. Now *AEDF* is a $\square$ (having its opposite sides $\parallel$). $\angle EDB = \angle C$ (ext.-int. $\angle$s) and hence $= \angle B$. $\therefore \triangle EBD$ is isosceles, and $ED = EB$. In like manner $FD = FC$. $\therefore AE + ED + DF + FA = AE + EB + FC + AF = AB + AC$.

Ex. 12. If from the diagonal BD of a square $ABCD$, BE be cut off equal to BC, and EF be drawn perpendicular to BD, show that DE is equal to EF, and also to FC.

In the $\triangle\ DEF$, $\angle\ DEF$ is a rt. $\angle$ (cons.), and $\angle\ EDF = \frac{1}{2}$ rt. $\angle$. $\therefore \angle\ EFD = \frac{1}{2}$ rt. $\angle$. $\therefore$ $\triangle\ DEF$ is isosceles, and $ED = EF$.

Join BF. Now $BE = BC$ (cons.) and BF is common to the $\triangle$s BEF and BCF. $\therefore \triangle\ BEF = \triangle\ BCF$ (§ 109); $\therefore EF = FC$; $\therefore DE = FC$.

Ex. 13. Show that the three lines drawn from the vertices of a triangle to the middle points of the opposite sides meet in a point.

Let ABC be the $\triangle$, AD, BF, and CE lines drawn from the vertices of the $\triangle$ to the middle points of the opposite sides. Complete the $\square$ $AHCB$, and draw AA' to the middle point of HC; CC' to the middle point of AH; and draw the diagonal BH. Then BH bisects AC (§ 138). But BF bisects AC (hyp.), $\therefore$ BH and BF, having two points in common, coincide. Now the lines AD and CC' trisect BH, also AA' and CE trisect BH (Ex. 10). $\therefore$ AD and CE intersect BF at the point O. $\therefore$ AD, BF, and CE meet in a point.

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Ex. 1. Show that, of all straight lines drawn from a point without a circle to the circumference, the least is that which, when produced, passes through the centre.

Let BEF be the ⊙, and A a point without the ⊙. Join A and C, the centre of the ⊙, intersecting the circumference at B. Draw any other line to the circumference as AD. Join DC. Now $AB + BC < AD + DC$. Take away from this inequality the equals BC and DC, and $AB < AD$.

Ex. 2. Show that, of all straight lines drawn from a point within or without a circle to the circumference, the greatest is that which meets the circumference after passing through the centre.

Let A and B be two points, one without and the other within the ⊙, and let AD and BH be two lines drawn through the centre, C, and terminating in the circumference at D and H respectively. Draw any other two lines, as AE and BF, terminating in the circumference at E and F respectively. Join CE and CF. Now $AC + CE > AE$. ∴ $AC + CD > AE$, or $AD > AE$. In like manner it may be shown that $BH > BF$. ∴ AD and BH are the greatest lines that can be drawn from A and B respectively to the circumference of the ⊙.

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Ex. AB, a chord of a circle, is the base of an isosceles triangle whose vertex C is without the circle, and whose equal sides meet the circle in D and E. Show that CD is equal to CE.

Draw AE and BD. In the $\triangle$s ACE and BCD, $CA = CB$ (hyp.); $\angle C = \angle C$, and $\angle CAE = \angle CBD$. $\therefore \triangle ACE = \triangle BCD$ (§ 107). $\therefore CD = CE$.

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Ex. Show that the least chord that can be drawn through a given point in a circle is perpendicular to the diameter drawn through the point.

Let C be any point in the $\odot$ DAB, and DE the diameter drawn through C. Also let AB be a chord drawn through the point C and $\perp$ to DE. Draw any other chord through C, as FH; and from O, the centre of the $\odot$, draw $OK \perp$ to FH. Now $OK < OC$ (§ 52). $\therefore AB < FH$ (§ 185).

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Ex. 1. Find the locus of the centre of a circumference which passes through two given points.

Connect the two given points A and B by the straight line AB, and draw DE bisecting AB at

right angles. Then DE will contain every point in the plane $AEBD$ equally distant from A and B (§ 58), and will be the locus required.

Ex. 2. Find the locus of the centre of a circumference of given radius, tangent externally or internally to a given circumference.

Let m be the given radius and ABC the given circumference. Draw OA, the radius of the given circumference. From O as a centre, with a radius $= OA - m$, describe the circumference DEF. Also, from O as a centre, with a radius $= OA + m$, describe the circumference HKL. The circumference DEF will be the locus of the centres of all circumferences which have m for a radius, and are tangent internally to ABC; and HKL will be the locus of the centres of all circumferences which have m for a radius and are tangent externally to ABC.

Ex. 3. A straight line is drawn through a given point A, intersecting a given circumference at B and C. Find the locus of the middle point P of the intercepted chord BC.

Let F be the centre of the given circumference. Through F draw AF intersecting the given circumference at D and E. From F draw FP. $\angle FPA$ is a rt. $\angle$ (§ 183). About the $\triangle$ FPA circum-

scribe a ⊙ intersecting the given circumference at H and K. The arc HFK is the locus required. For, if through any point P' of this arc the line AP' be drawn intersecting the given circumference at B' and C', and FP' be drawn, $\angle FP'A$ is rt. $\angle$ (being inscribed in a semicircle). $\therefore FP$ bisects the chord $B'C'$ (§ 183).

If the point A be within the ⊙, the circumference described upon FA as a diameter will be the locus required.

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Ex. 1. If the sides of a pentagon, no two sides of which are parallel, be produced till they meet; show that the sum of all the angles at their points of intersection will be equal to two right angles.

Let $ABCDE$ be the pentagon, and let a, b, c, d, e, be the $\angle$s formed by producing the sides of the pentagon. The sum of all the $\angle$s of the five $\triangle$s exterior to the pentagon is ten rt. $\angle$s. But $\angle$s $aAB + bBC + cCD + dDE + eEA =$ four rt. $\angle$s, and $\angle\ aBA + bCB + cDC + dED + eAE =$ four rt. $\angle$s (§ 159). $\therefore \angle$s $a, b, c, d, e, = (10 - 8)$ rt. $\angle$s $=$ two rt. $\angle$s.

Ex. 2. Show that two chords which are equally distant from the centre of a circle are equal to each other; and of two chords, that which is nearer the centre is greater than the one more remote.

I. Let AB and AC be two chords at equal distances from the centre O of the ⊙. Draw OE and $OF \perp$ to AB and AC respectively, and join OA. The rt. $\triangle$ AOE and AOF are equal (§ 109). $\therefore AE = AF$. $\therefore 2\,AE = 2\,AF$; that is, $AB = AC$.

II. Let AD be another chord farther from the centre than AC. Draw $OH \perp$ to AD. OH will intersect the chord AC at K. $AK > AH$ (§ 52), still more is $KC > AH$. $\therefore AK + KC > 2\,AH$, or $AC > AD$.

Ex. 3. If through the angles of an isosceles triangle which has each of the angles at the base double of the third angle, and is inscribed in a circle, straight lines be drawn touching the circle; show that an isosceles triangle will be formed which has each of the angles at the base one-third of the angle at the vertex.

Let the isosceles $\triangle$ ABC have each of the $\angle$s ABC and ACB equal to twice $\angle$ BAC, and be inscribed in ⊙ BAC. Also let DF, DE, and EF be drawn tangent to the ⊙ at the points A, B, and C respectively. Since $\angle$ $ABC = 2 \angle A$, arc $AC = 2$ arc BC; also arc $AB = 2$ arc BC. Now $\angle$ D is measured by $\frac{1}{2}$ arc $AC + \frac{1}{2}$ arc $BC -$ $\frac{1}{2}$ arc AB (§ 210); $\therefore \angle$ D is measured by $\frac{1}{2}$ arc BC. In like manner $\angle$ F is measured by $\frac{1}{2}$ arc

BC. $\therefore \triangle DEF$ is isosceles (§ 114). Now $\angle E$ is measured by $\frac{1}{2}$ arc $AB + \frac{1}{2}$ arc $AC - \frac{1}{2}$ arc BC; or, since arc $AB = 2$ arc BC, and arc $AC = 2$ arc BC, $\angle E$ is measured by $\frac{3}{2}$ arc BC. $\therefore \angle E = 3 \angle D = 3 \angle F$.

Ex. 4. ADB is a semicircle of which the centre is C; and AEC is another semicircle on the diameter AC; AT is a common tangent to the two semicircles at the point A. Show that if from any point F, in the circumference of the first, a straight line FC be drawn to C, the part FK, cut off by the second semicircle, is equal to the perpendicular FH to the tangent AT.

Draw AK, and produce it to meet the circumference at M. $\angle AHF$ is a rt. $\angle$ (hyp.), and $\angle AKF$ is a rt. $\angle$, for it is the supplement of AKC, and AKC is a rt. $\angle$ (being inscribed in a semicircle). Now arc $AF =$ arc FM, for radius $CKF \perp$ to the chord AM bisects the arc AFM. Join AF. Then $\angle HAF = \angle FAK$ (being measured by halves of the equal arcs AF and FM). $\therefore \triangle HAF = \triangle KAF$ (§ 110). $\therefore FH = FK$.

Ex. 5. Show that the bisectors of the angles contained by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.

Let $ABFC$ be an inscribed quadrilateral, let AB

and CF produced meet at E, and let AC and BF produced meet at D. Also, let the bisectors ES and DM intersect at O, and let ES intersect arc BF at V; the side BF at L, and AC at N, and the bisector DM intersect the arc FC at W, the side FC at K, and AB at H. In the ▵ KOE and EOH, $\angle KEO$ is measured by $\frac{1}{2}CS - \frac{1}{2}FV$ (§ 210); and $\angle HEO$ is measured by $\frac{1}{2}AS - \frac{1}{2}BV$ (§ 210). But $\angle KEO = \angle HEO$ (hyp.). $\therefore \frac{1}{2}CS - \frac{1}{2}FV = \frac{1}{2}AS - \frac{1}{2}BV$. $\therefore \frac{1}{2}CS + \frac{1}{2}BV = \frac{1}{2}AS + \frac{1}{2}FV$. In the ▵ DON and DOL, $\angle ODN = \angle ODL$ (hyp.); $\angle OND$ is measured by $\frac{1}{2}AS + \frac{1}{2}CF + \frac{1}{2}FV$ (§ 208); $\angle DLO$ is measured by $\frac{1}{2}BV + \frac{1}{2}CF + \frac{1}{2}CS$. Now, since $\frac{1}{2}CF$ is common, and $\frac{1}{2}AS + \frac{1}{2}FV$ has been shown to be equal to $\frac{1}{2}CS + \frac{1}{2}BV$, $\angle OND = \angle OLD$. $\therefore \angle DON = \angle DOL$ (§ 100). $\therefore$ the bisectors DM and ES intersect at rt. ∠s.

Ex. 6. If a triangle ABC be formed by the intersection of three tangents to a circumference whose centre is O, two of which, AM and AN, are fixed, while the third, BC, touches the circumference at a variable point P; show that the perimeter of the triangle ABC is constant, and equal to $AM + AN$, or $2\,AM$. Also show that the angle BOC is constant.

I. $AM = AN$, $BM = BP$, and $CN = CP$

(§ 241). $\therefore AB + BP = AM$; and $AC + CP = AN$. $\therefore AB + BP + AC + CP = AM + AN = 2\ AM$. Since AM is constant, the perimeter of the $\triangle\ ABC$ is constant.

II. From O, the centre of the $\odot$, draw OB and OC. Join OM and ON. The rt. $\triangle\ OPC =$ rt. $\triangle\ ONC$ (§ 109). $\therefore \angle\ COP = \angle\ CON$. In like manner the rt. $\triangle\ OPB =$ rt. $\triangle\ OMB$. $\therefore$ $\angle\ POB = \angle\ BOM$. $\therefore \angle\ BOC = \frac{1}{2} \angle\ MON$. Since MON is constant, $\angle\ BOC$ is constant.

Ex. 7. AB is any chord and AC is tangent to a circle at A, CDE a line cutting the circumference in D and E and parallel to AB; show that the triangles ACD and EAD are mutually equiangular.

$\angle\ BAE = \angle\ AED$ (alt.-int. $\angle$s). $\angle\ CAD = \angle\ AED$ (each measured by $\frac{1}{2}$ arc AD). $\therefore \angle\ BAE = \angle\ CAD$. $\angle\ ACD$ is measured by $\frac{1}{2}$ (arc ABE − arc AD); or, since arc $BE =$ arc AD, $\angle\ ACD$ is measured by $\frac{1}{2}$ arc AB. And $\angle\ AEB$ is measured by $\frac{1}{2}$ arc AB, $\therefore \angle\ ACD = \angle\ AEB$. $\therefore$ the two $\triangle$s are mutually equiangular (§ 100).

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Ex. 1. Draw two concentric circles, such that the chords of the outer circle which touch the inner may be equal to the diameter of the inner circle.

Given the outer $\odot\ CFDK$. From point E, the

centre of the ⊙, draw radius $EF \perp$ to the diameter CD. Bisect $\angle FED$ by the radius EB. Draw $BH \parallel$ to CD intersecting EF at H. Now $\angle HBE = \angle HEB$. $\therefore EH = HB$. With EH as radius and E as centre, describe a ⊙. Then any chord of the larger ⊙, as AB, which is tangent to the inner ⊙, will be equal to the diameter of the inner ⊙. For, $2\,EH =$ diameter of inner ⊙ $= 2\,HB$ = chord AHB. And all chords equally distant from the centre of the ⊙ are equal.

Ex. 2. Given the base of a triangle, the vertical angle, and the length of the line drawn from the vertex to the middle point of the base: construct the triangle.

Let $M =$ base of △ required; $N =$ distance of middle point of base to vertex; and $\angle S = \angle$ at vertex. On $KB = M$ describe a segment KRB which will contain $\angle S$ (§ 243). From F, the middle point of KB, as a centre, with a radius $= N$, describe an arc intersecting arc KRB at X. Draw XK and XB. The △ KXB will be the △ required. For $KB = M$ (cons.), $FX = N$ (cons.), and $\angle KXB = \angle S$.

Ex. 3. Given a side of a triangle, its vertical angle, and the radius of the circumscribing circle: construct the triangle.

Let X be the side of the required $\triangle$, Y the radius of the circumscribing $\odot$, and Z the vertical $\angle$. Draw $AC = X$, and at E, the middle point of AC, erect a $\perp$. From A as a centre, with a radius $= Y$, describe an arc cutting the $\perp$ in D. From D as a centre, with a radius $= Y$, describe a $\odot$. The circumference of the $\odot$ will pass through A and C (§ 59). At A construct an $\angle = \angle Z$, and produce the side until it meets the circumference in B. Join BC. $\triangle$ ABC is the $\triangle$ required.

Ex. 4. Given the base, vertical angle, and the perpendicular from the extremity of the base to the opposite side: construct the triangle.

Let N be the given base, P the given $\angle$, and M the given $\perp$. On $AB = N$ construct a segment AHB which shall contain the given $\angle$ P (§ 243). From A as a centre, with a radius $= M$, describe a $\odot$. Draw BG tangent to this $\odot$ and cutting arc of the segment which contains the given $\angle$ at G. Join GA. AGB is the $\triangle$ required.

Ex. 5. Describe a circle cutting the sides of a given square, so that its circumference may be divided at the points of intersection into eight equal arcs.

Let $ABCD$ be the given square. Draw GH

and FE ⊥ to the sides AB and BC respectively, at their middle points, and let O be the point of intersection of GH and FE. Then $BGOF$ is a square, and the diagonal BO bisects the rt. ∠ GOF. Bisect ∠s BOF and BOG by the lines OM and OK meeting the sides of the square at M and K. From O as a centre, with a radius = OK, describe a ⊙. It will be the ⊙ required. For side OK = side OM (being homologous sides of the equal △s OKB and OMB), and ∠ KOM is $\frac{1}{2}$ a rt. ∠, and ∴ $\frac{1}{8}$ of the angular magnitude about the point O. ∴ KM is $\frac{1}{8}$ of the circumference. In like manner each of the other arcs may be shown to be $\frac{1}{8}$ of the circumference.

Ex. 6. Construct an angle of 60°, one of 30°, one of 120°, one of 150°, one of 45°, and one of 135°.

Upon any given line construct an equilateral △; each ∠ of this △ will be an ∠ of 60°. Bisect one of these ∠s and each part will be an ∠ of 30°. Construct the equilateral △ ABC, and upon AB construct the equilateral △ ADB. Then ∠ DBC = 120°. Construct rt. ∠ ABC; on AB construct the equilateral △ ADB. ∠ DBC = 150°. Bisect a rt. ∠ and each part will be an ∠ of 45°. Draw AB ⊥ to DC. Bisect ∠ ABC by the line BE. ∠ DBE = 135°.

Ex. 7. In a given triangle ABC, draw QDE parallel to the base BC and meeting the sides of the triangle at D and E, so that DE shall be equal to $DB + EC$.

Draw BO and CO bisecting the $\angle$s B and C, respectively, and intersecting at O. Through O draw $DE \parallel$ to BC. Then DE is the required line. For, in $\triangle\ DBO$, $\angle\ DOB = \angle\ OBC$ (alt.-int. $\angle$s); $\angle\ DBO = \angle\ OBC$ (cons.) $\therefore \angle\ DOB = \angle\ DBO$. $\therefore \triangle\ DBO$ is isosceles, and $DO = DB$. In like manner we may prove $EO = EC$. $\therefore DE = DB + EC$.

Ex. 8. Given two perpendiculars, AB and CD, intersecting in O, and a straight line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF. (Two solutions.)

Bisect $\angle\ COB$, and produce bisector until it intersects EF at X. From X draw XP and XY $\perp$ to AB and CD respectively. Then $XYOP$ is the square required. For, in the rt. $\triangle\ OPX$, $\angle\ XOP$ is $\frac{1}{2}$ rt. $\angle$ (cons.). $\therefore$ its complement $\angle\ OXP$ is $\frac{1}{2}$ rt. $\angle$. That is, $\angle\ XOP = \angle\ OXP$. $\therefore\ OP = XP$. $\therefore\ XYOP$ is a square. In like manner bisect $\angle\ BOD$, and produce bisector until

it meets EF at M. Draw MN and $MS \perp$ to AB and CD respectively. Then $MNOS$ is the other square required.

Ex. 9. In a given rhombus to inscribe a square.

Let $ABCD$ be the given rhombus. Draw the diagonals AC and BD intersecting at O. Then the $\angle$s about O are rt. $\angle$s (§ 139). Bisect the $\angle$s at O, and let the bisectors meet the sides AB and CD at E and H, and AD and BC at G and F respectively. Draw EG, GH, HF, and FE. Then $EGHF$ is the square required. For, $\triangle$s ABC and ADC are equal (§ 133) and isosceles (§ 128). $\therefore$ $\angle$s BAC, BCA, DAC, DCA are equal. Now $\triangle\, FOC = \triangle\, HOC$, having OC common, $\angle\, FOC = \angle\, HOC$, and $\angle\, FCO = \angle\, HCO$. $\therefore$ $OF = OH$. Also, $\triangle\, FOC = \triangle\, AOG$; having $OC = OA$ (§ 138), $\angle\, FOC = \angle\, AOG$, and $\angle\, FCO = \angle\, OAG$. $\therefore$ $OF = OG$. In like manner $OE = OG$. That is, OF, OH, OG, OE are equal. Likewise $\angle$s FOH, HOG, GOE, EOF are rt. $\angle$s (Ex. 6, p. 23). $\therefore$ $\triangle$s FOH, HOG, GOE, EOF are all equal right isosceles $\triangle$s. $\therefore$ FH, HG, GE, EF are equal, and $\angle$s FHG, HGE, GEF, EFH are rt. $\angle$s (each being composed of two halves of a rt. $\angle$). That is, $EGHF$ is a square.

Ex. 10. If the base and vertical angle of a triangle be given; find the locus of the vertex.

Let BC be the base and M the vertical $\angle$ of the $\triangle$. On BC describe a segment BAC that will contain the given $\angle$ (§ 243). Then from any point A in the arc BAC draw AB and AC. $\angle BAC$ will equal the given $\angle$. $\therefore$ arc BAC will be the locus required.

Ex. 11. If a ladder, whose foot rests on a horizontal plane and top against a vertical wall, slip down; find the locus of its middle point.

Let BA be the position of the ladder when upright, BC when horizontal, and DE in any other position. Let O, P, and F be the centre of the ladder in these three positions respectively. From F draw $FM \parallel$ to CB and meeting BA at M. Since CB is $\perp$ to AB, FM is $\perp$ to AB. And since FM bisects DE, it bisects EB (p. 72, Ex. 9). $\therefore$ $\triangle EFM = \triangle BFM$ (§ 106.) $\therefore EF = FB$. That is, in any position the middle point of the ladder is at the distance of half the length of the ladder from B. $\therefore$ arc OFP, described from B as a centre with a radius $= BO$, is the locus required.

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Ex. 1. Show that the straight line which bisects the external vertical angle of an isosceles triangle is parallel to the base.

Let BAC be an isosceles Δ having the external-vertical $\angle\ CBD$ bisected by the line BE. The $\angle\ CBD = \angle A + \angle\ C$ (§ 105). $\therefore \angle\ EBC$, the half of $\angle\ DBC$, $= \angle\ C$. $\therefore BE$ is ∥ to AC (§ 69).

Ex. 2. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is drawn and terminated by the parallel straight lines. Show that the second straight line is bisected at the middle point of the first.

Let EF be a straight line terminated by the two ∥s AB and CD, and through its middle point O let a line be drawn meeting AB at G and CD at H. In $\triangle$ EOG and FOH, $EO = OF$ (cons.); $\angle\ GEO = \Delta\ OFH$ (alt.-int. $\angle$s); $\angle\ EOG = \angle\ HOF$ (vertical angles). $\therefore \Delta\ EOG = \Delta\ FOH$ (§ 107). $\therefore GO = HO$.

Ex. 3. Show that the angle between the bisector of the angle A of the triangle ABC and the perpendicular let fall from A on BC is equal to one-half the difference between the angles B and C.

In the $\Delta\ ABC$, let $\angle\ B$ be $> \angle\ C$, and let AD be a $\perp$ let fall from A on BC, and AE be the bisector of $\angle\ BAC$. Since ADB is a rt. $\angle$, $\angle\ B + \angle\ BAD =$ a rt. $\angle$. $\therefore \angle\ B =$ rt. $\angle - \angle\ BAD$. Likewise, $\angle\ C =$ rt. $\angle - \angle\ CAD$. $\therefore \angle B - \angle\ C = \angle\ CAD - \angle\ BAD$. But $\angle\ CAD = \angle\ CAE +$

$\angle EAD$, and $\angle BAD = \angle BAE - \angle EAD$. $\therefore$ $\angle CAD - \angle BAD = 2\angle EAD$. $\therefore \angle B - \angle C = 2\angle EAD$. $\therefore \angle EAD = \frac{1}{2}(\angle B - \angle C)$.

Ex. 4. In any right triangle show that the straight line drawn from the vertex of the right angle to the middle of the hypotenuse is equal to one-half the hypotenuse.

Let $\angle ACB$ be a rt. $\triangle$, and CE the line drawn from the vertex of the rt. $\angle$ C to the middle point of the hypotenuse AB. From A draw $AD \parallel$ to CB, and from B draw $BD \parallel$ to CA. Then $ACBD$ is a rectangle. Draw the diagonal CD. AB and CD bisect each other (§ 138). $\therefore$ CE and CD coincide. Now $\triangle ADC = \triangle ABC$ (§ 106). $\therefore \angle ACD = \angle BAC$. $\therefore EA = EC$ (§ 114).

Ex. 5. Two tangents are drawn to a circle at opposite extremities of a diameter, and cut off from a third tangent a portion AB. If C be the centre of the circle, show that ACB is a right angle.

Let XX' and YY' be two tangents drawn to the $\odot$ at extremities of diameter MK, cutting off from a third tangent AB. From C, the centre of the $\odot$, draw CH to the point of contact of AB with the $\odot$. The rt. $\triangle ACM =$ rt. $\triangle ACH$ (§ 109). $\therefore \angle ACM = \angle ACH$. The rt. $\triangle CKB =$ rt. $\triangle CHB$ (§ 109).

$\therefore \angle BCK = \angle BCH$. $\therefore \angle ACB = \frac{1}{2}$ the sum of the $\angle$s about the point C on the same side of the line MK. $\therefore \angle ACB$ is a rt. $\angle$.

Ex. 6. Show that the sum of the three perpendiculars from any point within an equilateral triangle to the sides is equal to the altitude of the triangle.

In the equilateral $\triangle ABC$, let OF, OH, and OE be $\perp$s drawn from any point O within the $\triangle$, to the sides AC, BC, and AB respectively, and let AX be the altitude of the $\triangle$. Through O draw $KM \parallel$ to BC, meeting AB at K, AC at M, and intersecting AX at Y. Draw $OP \parallel$ to BA. Then $OH = YX$ (§ 135). Draw $MN \perp$ to AK, meeting AK at N, and intersecting OP at S. Now $MN = AY$ (being altitudes of the equilateral $\triangle AKM$). $OE = SN$ (§ 135), and $OF = MS$ (being altitudes of the equilateral $\triangle POM$). $\therefore OE + OF = MN$. But $MN = AY$, $\therefore OE + OF = AY$; and it has been shown that $OH = YX$. $\therefore OE + OF + OH = AY + YX = AX$.

Ex. 8. Show that the angle contained by two tangents at the extremities of a chord is twice the angle contained by the chord and the diameter drawn from either extremity of the chord.

In the $\odot$ whose centre is O, draw the tangents AB and CB at the extremities of the chord AC,

and draw the diameter COD. Join OA, and draw OB intersecting AC at E. In the rt. $\triangle$ OAB and OCB, $OB = OB$ and $BA = BC$ (§ 241). $\therefore$ $\triangle\ OAB = \triangle\ OCB$ (§ 109). $\therefore \angle\ AOB = \angle\ COB$, and $\angle\ ABO = \angle\ CBO$. Now $\angle\ OEC$ is rt. $\angle$ (§ 113). In rt. $\triangle$ OEC and OCB, $\angle\ EOC$ is common. $\therefore \angle\ ECO = \angle\ EBC$. But $\angle\ EBC = \frac{1}{2} \angle\ ABC$. $\therefore \angle\ ABC = 2 \angle\ ACO$.

Ex. 9. If a circle can be inscribed in a quadrilateral; show that the sum of two opposite sides of the quadrilateral is equal to the sum of the other two sides.

Let $ABCD$ be a quadrilateral, and let the $\odot$ inscribed touch the sides at E, F, G, and H respectively. Then $AH + HD + BF + FC = AE + DG + EB + CG$ (§ 241), or $AD + BC = AB + DC$.

Ex. 10. If the sum of two opposite sides of a quadrilateral be equal to the sum of the other two sides; show that a circle can be inscribed in the quadrilateral.

In the quadrilateral $ABCD$, let $AB + DC = DA + CB$. Bisect $\angle$s DCB and ABC. Let these bisectors meet at some point O. Since O is in the bisectors of the $\angle$s DCB and ABC, it is equidistant from DC, CB, and BA. $\therefore$ a $\odot$ described from O

as a centre, with a radius equal to the distance from O to the side BA, will be tangent to the three sides DC, CB, and BA. If it be not tangent to AD, draw AE tangent to the ⊙. Then $AE + CB = AB + CE$ (Ex. 9). But $AD + CB = AB + CD$. By subtraction $AE - AD = DE$; that is, one side of a $\triangle$ is equal to the difference of the other two sides, which is absurd (§ 97). $\therefore$ the ⊙ is tangent to the side AD.

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Ex. Show that, if three or more non-parallel straight lines divide two parallels proportionally, they pass through a common point.

Let AA', BB' CC', DD' divide the parallel lines AD and $A'D'$ proportionally, so that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$. Let AA' and BB' meet in some point O. Join OC' and produce OC' until it cuts AD at some point H. Then $\frac{BH}{B'C'} = \frac{AB}{A'B'}$ (§ 288). But $\frac{BC}{B'C'} = \frac{AB}{A'B'}$ (hyp.). $\therefore \frac{BC}{B'C'} = \frac{BH}{B'C'}$. $\therefore BH = BC$. $\therefore$ H coincides with C. $\therefore$ CC' passes through O. In like manner DD' is shown to pass through O.

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Ex. 1. If any two straight lines be cut by parallel lines, show that the corresponding segments are proportional.

I. When the two straight lines AD and HG are ∥. Then the corresponding segments BC and EF, CD and FG are equal (§ 135). $\therefore BC : CD :: EF : FG$.

II. When AD and HG are not ∥. They will, when produced, meet at some point O. Then $OC : OF :: BC : EF :: CD : FG$ (§ 275). $\therefore BC : CD :: EF : FG$.

Ex. 2. If the four sides of any quadrilateral be bisected, show that the lines joining the points of bisection will form a parallelogram.

Let $ABCD$ be any quadrilateral, and E, F, H, K the middle points of the sides. Join EF, FH, HK, and KE, and draw the diagonals AC and BD. In the $\triangle\ ABD$, $AE = EB$ and $AK = KD$ (hyp.). $\therefore EK$ is ∥ to BD (§ 275). In $\triangle\ BCD$, $BF = FC$ and $DH = HC$. $\therefore FH$ is ∥ to BD, and $\therefore$ ∥ to EK. In like manner we may prove EF ∥ to HK. $\therefore EFHK$ is a ▱.

Ex. 3. Two circles intersect; the line $AHKB$ joining their centres A, B, meets them in H, K. On AB is described an equilateral triangle ABC, whose sides BC, AC, intersect the circles in F, E.

FE produced meets *BA* produced in *P*. Show that as *PA* is to *PK* so is *CF* to *CE*, and so also is *PH* to *PB*.

Join *EK* and *HF*, also draw *HM* and *AO* ∥ to *BC*. Since △ *ABC* is equilateral, ∠ *A* is $\frac{2}{3}$ of a rt. ∠; and since △ *AEK* is isosceles (*AE* and *AK* being radii), ∠ $AKE = \frac{2}{3}$ rt. ∠. ∴ △ *AKE* is equilateral. Also, ∠ *AKE* = ∠ *B*, ∴ *KE* is ∥ to *BC*. In like manner △ *BFH* is equilateral. ∴ ∠ *BHF* = ∠ *BAC*. ∴ *HF* is ∥ to *AC*. Now in the similar ⩟ *PAO* and *PKE* (similar by § 285), $PA : PK :: AO : KE$; or (since $AE = KE$, being sides of equilateral △ *AEK*) $PA : PK :: AO : AE$. Again, in the similar ⩟ *ECF* and *EAO* (similar since ∠ *CEF* = vertical ∠ *AEO*, and ∠ *EFC* = alt.-int. ∠ *EOA*), $AO : AE :: CF : CE$. ∴ $PA : PK :: CF : CE$.

In the similar ⩟ *PHM* and *PBF*, $PH : PB :: HM : BF$; or (since $HF = BF$ being sides of the equilateral △ *HBF*) $PH : PB :: HM : HF$. Again in the similar ⩟ *HMF* and *ECF* (similar since ∠ *MHF* = ∠ *C* (§ 77), and ∠ *HMF* = alt.-int. ∠ *EFC*), $HM : HF :: CF : CE$. ∴ $PH : PB :: CF : CE$. ∴ $PA : PK :: PH : PB$.

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Ex. If two circles are tangent internally, show that chords of the greater, drawn from the point of

tangency, are divided proportionally by the circumference of the less.

Let the two Ⓢ be tangent to each other at B, and let the chords BC and BA intersect the smaller circumference at E and D respectively. From B draw BH through the centre of the greater ⊙, intersecting the smaller circumference at G. Join EG, GD, CH, HA. BH passes through the centre of the smaller ⊙ (§ 189). ∆s BEG, BCH, BDG, and BAH are all rt. ∆s (being inscribed in semicircles), $\therefore$ EG is $\parallel$ to CH, and DG is $\parallel$ to AH (§ 72). $\therefore \frac{BE}{EC} = \left(\frac{BG}{GH}\right) = \frac{BD}{DA}$ (§ 275). That is, the chords BC and BA are divided at E and D proportionally.

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Note.—A common tangent is exterior when it touches the circles on the same side of a line joining their centres. A common tangent is interior when it intersects the line joining their centres between the two circles.

Ex. To draw a common tangent to two given circles:

I. When the common tangent is exterior.

II. When the common tangent is interior.

I. Let O and O' be the centres of the given ⊙, and let the radius of the first be the greater. With the centre O and a radius OC = the *difference* of the given radii, describe a circumfer-

ence. From O' draw a tangent $O'C$ to this circumference (§ 240). Join OC, and produce it to meet the given circumference at A. Draw $O'A' \parallel$ to OA, meeting the circumference in A', and join AA'. AA' is the common tangent to the two given ⓢ. For $OC = OA - O'A'$ (cons.), and $OC = OA - CA$. $\therefore CA = O'A'$. $\therefore ACO'A'$ is a $\square$ (§ 136). But $\angle OCO'$ is a rt. $\angle$(§ 187), $\therefore$ its supplement ACO' is a rt. $\angle$. $\therefore OCO'A'$ is a rectangle. $\therefore$ ⦣s A and A' are rt. ⦣s. $\therefore AA'$ is a tangent to both ⓢ. In like manner another exterior tangent BB' can be drawn.

II. With a centre O and a radius $OC =$ to the *sum* of the given radii, describe a circumference; and from O' draw a tangent $O'C$ to this circumference (§ 240). Join OC intersecting the given circumference at A. Draw $O'A' \parallel$ to CO, meeting the circumference at A', and join AA'. Now since $OC = OA + O'A'$, $AC = O'A'$. $\therefore ACO'A'$ is a $\square$ (§ 136). Since $\angle CAA'$ is a rt. $\angle$, $ACO'A'$ is rectangle. $\therefore$ ⦣s A and A' are rt. ⦣s. $\therefore AA'$ is a tangent to both the given ⓢ. In like manner another interior tangent BB' can be drawn.

Scholium. — If the given ⓢ intersect each other, only exterior tangents can be drawn. If they are tangent to each other externally, two exterior tangents are possible, but only one interior common tangent can be drawn. If they are tangent inter-

nally, only one exterior common tangent can be drawn, and the interior common tangents are impossible. If one circle is wholly within the other they can have no common tangent.

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Ex. 1. To divide a given line harmonically in a given ratio.

Let AB be the given line, and $m:n$ the given ratio. Upon the indefinite line AX take $AE = m$. On EX take $EF = n$, and on EA take $EH = n$. Join FB and HB, and draw $EC \parallel$ to FB meeting AB at C. Also from E draw a line $\parallel$ to HB meeting AB produced at D. Then $\frac{AE}{FE} = \frac{AC}{BC}$, and $\frac{AE}{HE} = \frac{AD}{BD}$ (§ 275). But $\frac{AE}{FE} = \frac{AE}{HE} = \frac{m}{n}$. $\therefore \frac{AC}{BC} = \frac{AD}{BD}$. $\therefore AB$ is divided harmonically at C and D in the given ratio.

Ex. 2. To find the locus of all the points whose distances from two given points are in a given ratio.

Let A and B be the given points, and let the given ratio be $m:n$. Suppose P to be a point of the required locus. Divide AB internally at C, and externally at D, in the ratio $m:n$. Draw PA, PB, PC, and PD. Produce AP to E. By the given

condition, $\frac{PA}{PB} = \frac{m}{n} = \frac{CA}{CB} = \frac{DA}{DB}$. Since $\frac{CA}{CB} = \frac{PA}{PB}$, PC is the bisector of $\angle APB$ (§ 309). From B draw a line ∥ to DP intersecting AP at F. Then $DA : DB :: PA : PF$ (§ 275). But $DA : DB :: PA : PB$ (hyp.). $\therefore PB = PF$. Hence $\angle PBF = \angle PFB$ (§ 112). Now $\angle PBF = \angle BPD$ (alt.-int. $\angle$s), and $\angle PFB = \angle DPE$ (ext.-int. $\angle$s). $\therefore \angle BPD = \angle DPE$. $\therefore PD$ bisects $\angle BPE$. The two bisectors PC and PD are $\perp$ to each other, $\therefore P$ is a point on the circumference having CD for a diameter. $\therefore$ this circumference is the locus required.

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Ex. 1. *ABC* is a triangle inscribed in a ⊙, and *BD* is drawn to meet the tangent to the circle at *A* in *D*, at an angle *ABD* equal to the angle *ABC*; show that *AC* is a fourth proportional to the lines *BD*, *AD*, *AB*.

In the $\triangle$ *ABC* and *ABD*, $\angle ABC = \angle ABD$ (cons.), and $\angle C = \angle DAB$ (each being measured by $\frac{1}{2}$ arc *AB*). $\therefore \triangle$ *ABC* and *ABD* are similar (§ 280). $\therefore BD : AD :: AB : AC$.

Ex. 2. Show that either of the sides of an isosceles triangle is a mean proportional between the

base and the half of the segment of the base, produced if necessary, which is cut off by a straight line drawn from the vertex at right angles to one of the equal sides.

Let ABC be an isosceles $\triangle$. From A draw a line $\perp$ to BA to meet BC produced at D. Bisect BD at E. Join AE. Then $AE = \frac{1}{2}BD = BE$ (Ex. 4, p. 138). $\therefore$ $\triangle$ EAB is isosceles. $\therefore \angle EAB = \angle B$. But $\angle ACB = \angle B$ (hyp.). $\therefore$ $\triangle$s EAB and ABC are similar (§ 280). $\therefore BC : BA :: BA : BE$.

Ex. 3. AB is the diameter of a circle, D any point in the circumference, and C the middle point of the arc AD. If AC, AD, BC be joined, and AD cut BC in E, show that the circle circumscribed about the triangle AEB will touch chord AC, and its diameter will be a third proportional to BC and AB.

Let AF be the diameter of the $\odot$ circumscribed about the $\triangle$ AEB. Join FB. $\angle CAD$ is measured by $\frac{1}{2}$ arc CD. $\angle ABC$ is measured by $\frac{1}{2}$ arc AC. Since the arcs CD and AC are equal (hyp.), $\angle CAD = \angle ABC$. But $\angle ABC$ is also measured by $\frac{1}{2}$ arc AE, $\therefore$ $\angle CAD$ is measured by $\frac{1}{2}$ arc AE. Now $\angle$s ABF and ACB are rt. $\angle$s (inscribed in semicircles), and $\angle F$ is measured by $\frac{1}{2}$ arc AEB. Also $\angle CAB$, which is equal to $\angle CAE + \angle EAB$,

is measured by $\frac{1}{2}$ (arc EA + arc EB). $\therefore \angle CAB = \angle F$. $\therefore$ $\triangle$ ABC and ABF are similar (§ 281). But $\angle BAF$ is the complement of $\angle F$. $\therefore \angle CAB + \angle BAF =$ rt. $\angle$. Hence CA being $\perp$ to the diameter AF at A, is tangent to the $\odot$ AEB. Also, in the similar $\triangle$ ABC and ABF, $BC : AB :: AB : AF$; that is, AF is the third proportional to BC and AB.

Ex. 4. From the obtuse angle of a triangle draw a line to the base, which shall be a mean proportional between the segments into which it divides the base.

Let B be the obtuse $\angle$ of the $\triangle$ ABC. Circumscribe a $\odot$ about the $\triangle$ ABC, and draw the diameter BE passing through the centre D of the $\odot$. On BD as a diameter describe a semicircle intersecting AC at K. Draw BK, and produce it to meet the circumference at H. BK is the line required. For, join DK and EH. Then $\angle BKD = \angle BHE$ (each being inscribed in a semicircle). $\therefore$ KD is $\parallel$ to HE. $\therefore BK : KH :: BD : DE$. Since $BD = DE$, $BK = KH$. Moreover, $AK : BK :: KH : KC$ (§ 290). Substitute for KH its equal BK; then $AK : BK :: BK : KC$.

Ex. 5. Find the point in the base produced of a right triangle, from which the line drawn to the

angle opposite to the base shall have the same ratio to the base produced which the perpendicular has to the base itself.

Let ABC be a rt. △, CB its base, and CA its altitude. From C as a centre, with a radius $= CA$, describe an arc cutting AB at D, and draw CD. From A draw a line ∥ to DC meeting BC produced at E. Then $EA : EB :: CD : CB$ (§ 275). Substitute for CD its equal AC, and we have $EA : EB :: CA : CB$.

Ex. 6. A line touching two circles cuts another line joining their centres; show that the segments of the latter will be to each other as the diameters of the circles.

Let BA be the line joining the centres of the two ⓢ, and CD the line touching the ⓢ at C and D respectively and intersecting the line AB at E. Join BC and AD. In the rt. ⨺ ECB and EDA, $\angle CEB = \angle AED$; ∴ ⨺ CEB and AED are similar (§ 281), ∴ $EB : EA :: BC : AD$, or, $:: 2BC : 2AD$.

Ex. 7. Required the locus of the middle points of all the chords of a circle which pass through a fixed point.

Let A be the centre of the given ⊙ and B the fixed point. Let any chord of the first ⊙ be drawn,

so that, produced if necessary, it may pass through B. Let P be the middle point of this chord. Then P is a point of the required locus. Join AP. AP is $\perp$ to the chord of which P is the middle point (§ 184). $\therefore P$ is on the circumference of a $\odot$ described on AB as a diameter. If B be within the given $\odot$, the locus is the circumference of the $\odot$ described on AB as a diameter. If B be without the given $\odot$, the locus is that part of the circumference of the $\odot$ described on AB as a diameter which is within the given $\odot$.

Ex. 8. O is a fixed point from which any straight line is drawn meeting a fixed straight line at P; in OP a point Q is taken such that OQ is to OP in a fixed ratio. Determine the locus of Q.

From O draw a line $\perp$ to the fixed straight line and meeting it at C. In OC take a point D, such that $OD : OC =$ the given ratio. Join QD. Then $\triangle$s ODQ and OCP are similar (§ 284). $\therefore \angle ODQ = \angle OCP$. $\therefore \angle ODQ$ is a rt. $\angle$. Hence a straight line drawn through $D \perp$ to OD is the locus required.

Ex. 9. O is a fixed point from which any straight line is drawn meeting the circumference of a fixed circle at P; in OP a point Q is taken such that OQ is to OP in a fixed ratio. Determine the locus of Q.

Let C be the centre of the fixed ⊙. Join OC, and in OC take a point D such that $OD : OC =$ the fixed ratio. Join CP and DQ. Then ⚠ OCP and ODQ are similar (§ 277). $\therefore DQ : CP :: OD : OC$. That is $DQ : CP =$ the fixed ratio. Since CP is a radius of the given ⊙, CP is constant; $\therefore DQ$ is constant. Hence the circumference of a ⊙ of which D is the centre and DQ the radius is the locus required.

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Ex. 1. Given $x = \sqrt{5}$, $y = \sqrt{7}$, $z = 2\sqrt{3}$; to construct x, y, and z.

Construct as in § 360.

Ex. 2. Given $2 : x :: x : 3$.

$\therefore x = \sqrt{6}$. Construct as in § 360.

Ex. 3. Construct a square equivalent to a given hexagon.

Reduce the hexagon to a △ (§ 351), and find a mean proportional between the altitude and one-half the base (§ 306). On this mean proportional as a side, construct a square. This will be the square required.

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Ex. 1. Construct a square equivalent to the sum of three given squares whose sides are respectively 2, 3, and 5.

Take line m as the linear unit. On indefinite line AK, take $AB = 5\ m$. At A erect a $\perp$ $AC = 3\ m$. Join CB. At C erect $CD = 2\ m \perp$ to CB. Join DB. On DB construct a square. This will be the square required.

Ex. 2. Construct a square equivalent to the difference of two given squares whose sides are respectively 7 and 3.

Take line m as the linear unit. On indefinite line AK take $AB = 3\ m$. At A erect a $\perp$ to AB. From B as a centre, with a radius $= 7\ m$, describe an arc intersecting the $\perp$ at C. AC is a side of the square required.

Ex. 3. Construct a square equivalent to the sum of a given triangle and a given parallelogram.

Let P be the given $\triangle$, and Q the given $\square$. Find a square $P' \approx \triangle\ P$, by taking for its side a mean proportional between the base and one-half the altitude of the $\triangle$ (§ 306). Also find a square $Q' \approx$ to the given $\square$ (§ 354). Then find a square S' $\approx$ to the sum of the squares P' and Q' (§ 346). S' is the square required.

Ex. 4. Construct a rectangle having the difference of its base and altitude equal to a given line, and its area equivalent to the sum of a given triangle and a given pentagon.

Let a be the given line, A the given $\triangle$, B the given pentagon. Construct a square $\approx$ to A (§ 355), and let X be one of its sides. Also construct a square $\approx$ to B (§ 356), and let Y be one of its sides. Construct a square $\approx$ to the sum of the squares on X and Y (§ 346), and let Z be one of its sides. Take $MN = a$. On MN as a diameter describe a $\odot$. From M draw MS tangent to the $\odot$ and $= Z$. Through the centre of the $\odot$ draw SB intersecting the circumference at C and B. Construct the rectangle R, having SB for its base and SC for its altitude. R is the rectangle required (§ 359).

Ex. 5. Given a hexagon; to construct a similar hexagon whose area shall be to that of the given hexagon as 3 to 2.

Let P be the given hexagon; and let m be the linear unit of measure. On the indefinite line LR, take $LM = m$, $LN = 3\,m$, and $LO = 4\,m$. On LN and LO as diameters, describe semi-circumferences. At M erect $\perp$ MK intersecting the circumferences at H and K. Then $MH = \sqrt{2}$ and $MK = \sqrt{3}$ (§ 360). Draw MS making any convenient $\angle$ with LR. On MS take $ME = AB$, a side of the given hexagon. Join HE, and draw $KD \parallel$ to HE. On MD, homologous to AB, construct a polygon P' similar to P. P' is the hexagon required (§ 362).

Ex. 6. Construct a pentagon similar to a given pentagon and equivalent to a given trapezoid.

Let P be the given pentagon and Q the given trapezoid, and AB a side of the pentagon P. Find a square $\approx P$ (§ 356), and let x be one of its sides. Also find a square $\approx Q$, and let y be one of its sides. Then find a fourth proportional to x, y and AB (§ 304), and let the fourth proportional be $A'B'$. Upon $A'B'$, homologous to AB, construct polygon P' similar to given polygon P. P' is the polygon required (§ 361).

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Ex. 1. The perpendicular distance between two parallels is 30, and a line is drawn across them at an angle of $45°$; what is its length between the parallels?

Ans. 42.4+.

Ex. 2. Given an equilateral triangle each of whose sides is 20; find the altitude of the triangle, and its area.

Ans. Alt. 17.32+; area, 173.2+.

Ex. 3. Given the angle A of a triangle equal to $\frac{2}{3}$ of a right angle, the angle B equal to $\frac{1}{3}$ of a right angle, and the side a, opposite the angle A, equal to 10; construct the triangle.

$\angle A + \angle B =$ rt. $\angle$. $\therefore$ the third $\angle$ of the $\triangle$ $C =$ rt. $\angle$. Let line m be the unit of measure. Lay off indefinite line CK. At C erect the $\perp$ $CB = 10\,m$. At B construct an $\angle = \frac{1}{3}$ rt. $\angle$, and produce the side to meet CK at A. ACB is the $\triangle$ required.

Ex. 4. The two segments of a chord intersected by another chord are 6 and 5, and one segment of the other chord is 3; what is the other segment of the latter chord?

Let AC and DB be the two chords intersecting at K. Given $DK = 5$, and $KB = 6$, and $AK = 3$, it is required to find the value of KC. $AK : KB :: DK : KC$ (§ 290). $\therefore 3\,KC = 30$. $\therefore KC = 10$.

Ex. 5. If a circle be inscribed in a right triangle: show that the difference between the sum of the two sides containing the right angle and the hypotenuse is equal to the diameter of the circle.

Let ABC be the given rt. $\triangle$ having a $\odot$ inscribed within it. From K, the centre of the $\odot$, draw KD $\perp$ to the hypotenuse AB, $KH \perp$ to the base AC, and $KE \perp$ to BC. Then $AD = AH$ and $BD = BE$ (§ 241). Now $(AC + BC) - AB = EC + CH$. But $EC + CH = HK + KE$. For $KECH$ is a square, all its $\angle$s being rt. $\angle$s, and all its sides being equal. But $KH + KE =$ diameter of the $\odot$; $\therefore (AC + BC) - AB =$ diameter of the $\odot$.

Ex. 6. Construct a parallelogram the area and perimeter of which shall be respectively equal to the area and perimeter of a given triangle.

Let $A'B'C'$ be the given △. Construct a ▱ $DEBA$ ≎ to the the given △ $A'B'C'$, by finding a square ≎ to the given △ (§ 355), and then constructing a ▱ ≎ to this square. Draw the line $XY =$ the perimeter of the given △. Take $MS = AB$, base of ▱ $DEBA$. Draw $MF \perp$ to MS at M, and = altitude of ▱ $DEBA$. Draw $FN \parallel$ to MS. On XY take $XL = 2\ MS$. Then LY will be the sum of two sides of the required ▱. From M as a centre, with a radius $= \frac{1}{2}\ LY$, draw an arc intersecting FN at O; and from S as a centre, with the same radius, draw an arc intersecting FN at N. Join MO and SN. Then $MONS$ is the ▱ required. For, it is ≎ ▱ $DEBA$ (having an equal base and altitude) and hence ≎ to the given △; and its perimeter is equal to line XY which is equal to the perimeter of the △.

Ex. 7. Given the difference between the diagonal and side of a square; construct the square.

Let m be the given difference. Draw $DE = m$, and at E draw $EF \perp$ to DE and equal to it. Through D and F draw DFC, making $FC = EF$. At C erect $CB \perp$ to DC and meeting DE produced at B. Since $EF = DE$, △ DEF is an

isosceles rt. △. ∴ ∠ EDF is one-half a rt. ∠. Since ∠ DCB is a rt. ∠, and ∠ EDC is one-half a rt. ∠, ∠ CBD is one-half a rt. ∠. ∴ $CB = CD$ (§ 114). Complete the square $BADC$. Draw BF. In the ⊿ EFB and FBC, ∠ $E =$ ∠ C (each being a rt. ∠), FB is common, and $EF = FC$ (cons.). ∴ △ EFB = △ FBC. ∴ $EB = BC$. ∴ $DE = DB - BC$. ∴ $BADC$ is the square required.

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Ex. 1. Show that two triangles which have an angle of the one equal to the supplement of the angle of the other are to each other as the products of the sides including the supplementary angles.

Let ⊿ ABC and CDE have ∡ ACB and DCE supplements of each other. Place these ⊿ so that their bases may lie in the same straight line, and CD lie on the side CB. From D draw a line ∥ to BA, meeting AC at F. Then $\frac{\triangle FCD}{\triangle CDE} = \frac{FC}{CE}$ (§ 326). Multiply $\frac{FC}{CE}$ by $\frac{CD}{CD}$. Then $\frac{\triangle FCD}{\triangle CDE} = \frac{FC \times CD}{CE \times CD}$. But $\frac{\triangle ACB}{\triangle FCD} = \frac{AC \times CB}{FC \times CD}$ (§ 341). Multiply these last two equalities; $\frac{\triangle ACB}{\triangle CDE} = \frac{AC \times CB}{CE \times CD}$.

Ex. 2. Show, geometrically, that the square described upon the sum of two straight lines is equivalent to the sum of the squares described upon the two lines *plus* twice their rectangle.

Let AB and BC be any two lines, and AC their sum. On AC describe the square $ACDE$. On AE take $AF = AB$, and from F draw a line ‖ to AC meeting CD at X. From B draw a line ‖ to AE meeting ED at H and intersecting FX at K. $ABKF$ is a square described upon AB. Since $AF = AB$ (cons.), $KXDH$ is equal to the square on BC. For, $KX = BC$ and $DX = EF$ (‖s comprehended between ‖s). But $EF = BC$, since each is the difference between the side of the square EC and the side of the square FB. Now, since FK and KB are each equal to AB, and KX and KH are each equal to BC, the rectangles FH and KC are each equal to $AB \times BC$. $\therefore$ square on $AC \approxeq$ square on AB + square on BC + twice the rectangle $AB \times BC$.

Ex. 3. Show, geometrically, that the square described upon the difference of two straight lines is equivalent to the sum of the squares described upon the two lines *minus* twice their rectangle.

Let AB and BC be any two lines, and AC their difference. On AB describe the square $ABKF$, and upon AF, take $AE = AC$. From C draw CG

$\parallel$ to BK, meeting FK at G. Through E draw a line $\parallel$ to AB intersecting CG at D, and meeting BK at H. Produce HE to I, making $EI = CB$. Now $AB = AF$ and $AC = AE$, $\therefore CB = EF$. Complete the square $EFLI$. $GK = CB = LF$. $\therefore GL = FK = AB$. $\therefore$ rectangles $CBKG$ and $GLID$ are each equal to $AB \times BC$. If these two rectangles be taken from the whole figure $ABKLIE$, which is equal to the sum of the squares on AB and BC, there will be left the square $ACDE$.

Ex. 4. Show, geometrically, that the rectangle of the sum and difference of two straight lines is equivalent to the difference of the squares on those lines.

Let AB be the greater of two lines and AC the less. On AB describe the square $ABDE$; and on AC describe the square $ACD'E'$. Produce $E'D'$ to meet BD at F. Now the difference of the squares on AB and AC is the two rectangles EF and FC. The base of the rectangle EF is $ED = AB$; and the base of the rectangle AC is $D'C = AC$. $\therefore$ the sum of the two bases is $AB + AC$, and the common altitude is $CB = AB - AC$. $\therefore$ $\overline{AB}^2 - \overline{AC}^2 = (AB + AC)(AB - AC)$.

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Ex. 1. Show that an equilateral polygon circumscribed about a circle is regular if the number of its sides be *odd*.

Let the sides of the circumscribed equilateral polygon $ABCDE$ touch the circumference at S, K, M, N, and P. From the centre O draw OC, OD, OE, and OK, OM, ON, OP. OM, ON, and OP are $\perp$ to the sides of the polygon (§ 187). Rt. $\triangle\ OMD =$ rt. $\triangle\ OND$ (§ 109). In like manner, rt. $\triangle\ OCM =$ rt. $\triangle\ OCK$, and rt. $\triangle\ OEN$ $=$ rt. $\triangle\ OEP$. $\therefore \angle\ ODM = \angle\ ODN$, and $\angle\ OCN = \angle\ OCK$, and $\angle\ OEN = \angle\ OEP$. Revolve $\triangle\ ODE$ upon OD as an axis until it comes into the plane of $\triangle\ ODC$. Then DE will fall on DC, and point E will fall on C (since $DE = DC$, the polygon being equilateral). $\therefore \angle\ OED = \angle\ OCD$. $\therefore \angle\ E\ (= 2 \angle\ OED) = \angle\ C\ (= 2 \angle\ OCD)$. In like manner we may prove the other $\angle$s of the polygon equal. $\therefore$ the polygon is regular.

Ex. 2. Show that an equiangular polygon inscribed in a circle is regular if the number of its sides be *odd*.

Let $ABCDE$ be an inscribed equiangular polygon of an odd number of sides. From the centre of the $\odot$ draw the radii OA, OB, etc., and also draw Oa,

Ob, etc., $\perp$ to the sides of the polygon. Upon Oa as an axis revolve the quadrilateral $OaAE$ until it comes into the plane of the quadrilateral $OaBC$. Since Oa is perpendicular to the chord AB, $aA = aB$, and the point A will fall upon the point B. And, since $\angle A = \angle B$, AE will fall upon BC. Since point E falls in the line BC, and at the same time in arc BCD, it will fall at their point of intersection C. $\therefore AE = BC$. In like manner we may show $AB = CD$, $BC = DE$, $CD = EA$, $DE = AB$. $\therefore$ the polygon is regular.

Ex. 3. Show that *any* equiangular polygon circumscribed about a circle is regular.

Let ABC, etc., be an equiangular polygon circumscribed about a $\odot$. From the centre O draw OB, OC, etc., to the vertices of the polygon; and draw OD, OE, etc., to the points of tangency. Rt. $\triangle\ DOB =$ rt. $\triangle\ EOB$ (§ 109). $\therefore \angle DBO = \angle EBO$ and $BD = BE$. In like manner we may show that OC bisects $\angle C$. In rt. $\triangle$s BEO and CEO, $\angle EBO = \angle ECO$, and EO is common. $\therefore \triangle\ BEO = \triangle\ CEO$ (§ 111). $\therefore BE = EC$. $\therefore BD + DA = BE + EC$, or $BA = BC$. That is, the polygon is equilateral and regular.

Ex. 4. Show that the side of a circumscribed equilateral triangle is double the side of an inscribed equilateral triangle.

Let ABC be an equilateral inscribed $\triangle$, and through the points A, B, and C draw DE, DH, and EH tangents to the $\odot$, and intersecting at D, E, and H. $\angle DBA$ is an $\angle$ of 60°, being measured $\frac{1}{2}$ arc BA, which is $\frac{1}{3}$ of the circumference (§ 209). $\angle DAB$ is an $\angle$ of 60° (for the same reason). $\therefore \angle BDA = 60°$ (§ 98). $\therefore \triangle DBA$ is equiangular, and hence equilateral. $\therefore DA = BA$. In like manner we may prove $EA = CA$. $\therefore DE = 2\ AB$.

Ex. 5. Show that the area of a regular inscribed hexagon is three-fourths of that of the regular circumscribed hexagon.

Let O be the centre of the $\odot$, HK a side of the regular inscribed hexagon. Join OH, and draw $OC \perp$ to HK. OC bisects HK (§ 183). Now the circumscribed hexagon is to the inscribed hexagon, as $\overline{OH}^2$ is to $\overline{OC}^2$ (§ 344). $HK = OH$ (§ 391). $\therefore HC\ (= \frac{1}{2}\ HK) = \frac{1}{2}\ OH$. But $\overline{OH}^2 = \overline{OC}^2 + \overline{HC}^2$ (§ 331) $= \overline{OC}^2 + (\frac{1}{2}\ OH)^2 = \overline{OC}^2 + \frac{1}{4}\ \overline{OH}^2$. $\therefore \frac{3}{4}\ \overline{OH}^2 = \overline{OC}^2$. $\therefore$ the regular inscribed hexagon is equal to $\frac{3}{4}$ of the regular circumscribed hexagon.

Ex. 6. Show that the area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

Let $ABCDEF$ be a regular inscribed hexagon, and let BDF be the inscribed $\triangle$, and HGK be the regular circumscribed $\triangle$, touching the circumference at B, D, and F. From the centre O draw OB, OD, and OF. In $\triangle$s FAB and BOF, $FA = FO$, $BA = BO$, and FB is common; $\therefore$ $\triangle\ BAF = \triangle\ BOF$ (§ 108). In like manner, $\triangle\ BOD = \triangle\ BCD$, and $\triangle\ FOD = \triangle\ FED$. $\therefore$ area of hexagon is twice area of $\triangle\ BDF$. Since a side of the circumscribed equilateral $\triangle$ is twice the side of an inscribed equilateral $\triangle$ (Ex. 4), the area of the circumscribed $\triangle$ is four times the area of the inscribed $\triangle$ (§ 343). Hence, if we denote the area of the inscribed $\triangle$ by 1, the area of the regular inscribed hexagon will be 2, and of the circumscribed $\triangle$ will be 4, and they will give the proportion $1 : 2 :: 2 : 4$.

Ex. 7. Show that the area of a regular inscribed octagon is equal to that of a rectangle whose adjacent sides are equal to the sides of the inscribed and circumscribed squares.

Let $ABCD$ be the circumscribed square, $EFGH$ the inscribed square, and $EJFKGLHI$ the regular inscribed octagon. Through M, the centre of the $\odot$, draw the diameters HF, EG, LJ, and IK, and let LJ intersect EF at O. The $\triangle\ JMF = \frac{1}{8}$ area of octagon. EF is $\perp$ to JM; for J and M are at

equal distances from E and F; and $OF = EO$ (§ 60). $MJ \times OF = \frac{1}{4}(EF \times AB)$. For $MJ = \frac{1}{2} AB$, and $OF = \frac{1}{2} EF$. Now $\triangle JMF$ has the same base JM, and the same altitude OF, as rectangle $MJ \times OF$. $\therefore \triangle JMF = \frac{1}{2}$ rectangle $MJ \times OF$. $\therefore 8(\triangle JMF) = 4(MJ \times OF)$. $\therefore$ octagon EJF, etc., $=$ rectangle $EF \times AB$.

Ex. 8. Show that the area of a regular inscribed dodecagon is equal to three times the square on the radius.

Let $ABCD$, etc., be a regular inscribed dodecagon, and G the centre of the $\odot$. Draw GA, GB, GC, and join AC intersecting GB at F. Since GF bisects the $\angle$ at the vertex of the isosceles $\triangle AGC$, it is $\perp$ to AC at its middle point (§ 113). Now area of $\triangle GAC = AF \times GF$ (§ 324); and area of $\triangle ABC = AF \times BF$. $\therefore$ area of the figure $GABC = AF \times BG$. The area of the dodecagon $= 6\ GABC$, and $\therefore = 6(AF \times BG)$. Since AC is one side of a regular inscribed hexagon, it is equal to the radius of the $\odot$. $\therefore AF = \frac{1}{2}$ the radius $= \frac{1}{2} BG$. Hence area of the dodecagon $= 6(\frac{1}{2} BG \times BG) = 3\overline{BG}^2$.

Ex. 9. Given the diameter of a circle 50; find the area of the circle. Also, find the area of a sector of 80° of this circle.

Area of the ⊙ $= \pi R^2 = 1963.5$. The area of the sector of $80° = \dfrac{80}{360}$, or $\frac{2}{9}$ of the ⊙ $= 436.3+$.

Ex. 10. Three equal circles touch each other externally and thus inclose one acre of ground; find the radius in rods of each of these circles.

Let R represent the radius of the equal Ⓢ. Join the three centres, thus forming $\triangle ABC$. The lines joining the centres will pass through the points of contact (§ 189), and, hence, each side of the $\triangle = 2R$; $\therefore$ the triangle is equilateral and incloses the given area and three equal sectors. Since the $\angle$ of each sector $= \frac{1}{3}$ 2 rt. $\angle$s, the three sectors together $=$ a semicircle. The area of the semicircle is $\dfrac{\pi R^2}{2}$, $\therefore$ the area of the $\triangle = \dfrac{\pi R^2}{2} + 160$. But the area of the $\triangle$ also equals $R \times$ by the altitude, which is $R\sqrt{3}$. $\therefore R^2\sqrt{3} = \dfrac{\pi R^2}{2} + 160$. $\therefore R^2(2\sqrt{3} - \pi) = 320$; or, $R^2 = \dfrac{320}{2\sqrt{3} - 3.1416} = 992.24$. $\therefore R = 31.48+$.

Ex. 11. Show that in two circles of different radii, angles at the centres subtended by arcs of equal length are to each other inversely as the radii.

Let A and A' be the two angles, b and b' the

corresponding arcs, R and R' the radii of the corresponding ⓢ. Then $\frac{A}{360} = \frac{b}{2\pi R}$ and $\frac{A'}{360} = \frac{b'}{2\pi R'}$ (§§ 202 and 376). $\therefore \frac{A}{360} : \frac{A'}{360} = \frac{b}{2\pi R} : \frac{b'}{2\pi R'}$, or $A : A' = \frac{b}{R} : \frac{b'}{R'}$. But $b = b'$ (hyp.). $\therefore A : A' = R : R'$.

Ex. 12. Show that the square on the side of a regular inscribed pentagon, minus the square on the side of a regular inscribed decagon, is equal to the square on the radius.

Let AB and AC be the sides of a regular pentagon and decagon inscribed in a ⊙ whose centre is O. Join OA, OB, OC, BC. Draw OD bisecting $\angle AOC$, and meeting AB at D, and join DC. Now $\triangle AOD = \triangle COD$ (having $OA = OC$, OD common, and $\angle AOD = \angle COD$). $\therefore DA = DC$ and $\triangle ADC$ is isosceles. But $\triangle ACB$ is isosceles, and has one of its equal ∡, namely, BAC, in common with $\triangle ADC$; $\therefore$ ▵ ACB and ADC are similar. Again, $\angle AOB = \frac{4}{5}$ rt. $\angle$. $\therefore \angle DBO = \frac{3}{5}$ rt. $\angle$, and $\angle DOB = \angle DOC + \angle COB = \frac{3}{5}$ rt. $\angle$. $\therefore$ $\triangle BDO$ is isosceles, and similar to $\triangle AOB$. In the similar ▵, ACB and ADC, we have $AB : AC :: AC : AD$; that is, $\overline{AC}^2 = AB \times AD$; and in the similar ▵, AOB and BDO, $AB : OB :: OB : BD$;

that is, $\overline{OB}^2 = AB \times BD$; $\therefore \overline{AC}^2 + \overline{OB}^2 = AB \times (AD + BD) = \overline{AB}^2$. $\therefore \overline{AB}^2 - \overline{AC}^2 = \overline{OB}^2$.

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Ex. Let R denote the radius of a regular inscribed polygon, r the apothegm, a one side, A one angle, and C the angle at the centre; show that

1. In a regular inscribed triangle $a = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$.

Let ADE be a regular inscribed $\triangle$. Since the $\triangle$ is regular, it is equilateral and equiangular. $\therefore$ $\angle A$ is $\frac{1}{3}$ of 2 rt. $\angle$s, or 60°.

Since AD is a chord of $\frac{1}{3}$ of the circumference, $\angle ACD$ is measured by $\frac{1}{3}$ of the circumference, $\therefore \angle C = 120°$.

Draw $CO \perp$ to AD, and produce it to meet the circumference at M. Join AM and MD. Then AM is a side of a regular inscribed hexagon. $\therefore$ $AM = AC$. $\therefore$ the rt. $\triangle AOM =$ rt. $\triangle AOC$ (§ 109). $\therefore CO = OM$. $\therefore CO = \frac{1}{2}R$. But $CO = r$, $\therefore r = \frac{1}{2}R$.

In the rt. $\triangle AOC$, $\overline{AO}^2 = \overline{AC}^2 - \overline{OC}^2 = \overline{AC}^2 - \frac{1}{4}\overline{AC}^2$. $\therefore \overline{AO}^2 = \frac{3}{4}\overline{AC}^2$. But $\overline{AO}^2 = \frac{1}{4}\overline{AD}^2$; $\therefore \overline{AD}^2 = 3\,\overline{AC}^2$. $\therefore AD = AC\sqrt{3}$. That is, $a = R\sqrt{3}$.

2. In an inscribed square $a = R\sqrt{2}$, $r = \frac{1}{2}R\sqrt{2}$, $A = 90°$, $C = 90°$.

Let $ABOD$ be an inscribed square. From the centre C, draw CD and CO, also draw $CA \perp$ to DO. Since $\angle\ DCO$ is measured by DO, $\frac{1}{4}$ the circumference, it is a rt. $\angle$. Since $\angle\ ADO$ is an $\angle$ of a square, $\angle\ ADO$ is a rt. $\angle$. In the rt. $\angle\ DCO$, $\overline{DO}^2 = \overline{CD}^2 + \overline{CO}^2 = 2\ \overline{CD}^2$. $\therefore\ DO = R\sqrt{2}$; that is $a = R\sqrt{2}$. $DA = \frac{1}{2} DO$; $\angle\ CDO = \frac{1}{2}$ rt. $\angle$; $\therefore \angle\ DCA = \frac{1}{2}$ rt. $\angle$. $\therefore DA = CA$ (§ 114). $\therefore\ CA = \frac{1}{2} DO = \frac{1}{2} R\sqrt{2}$. That is, $r = \frac{1}{2} R\sqrt{2}$.

3. In a regular inscribed hexagon $a = R$, $r = \frac{1}{2} R\sqrt{3}$, $a = 120°$, $C = 60°$.

Let AB be a side of a regular inscribed hexagon, C the centre of the $\odot$. Join CA and CB, and draw $CH \perp$ to AB. $AB = R$ (§ 391). That is, $a = R$.

Since CH bisects AB, $\overline{CH}^2 = \overline{CA}^2 - \frac{1}{4}\overline{CA}^2$. $\therefore \overline{CH}^2 = \frac{3}{4}\overline{CA}^2$. $\therefore CH = \frac{1}{2} CA\sqrt{3}$. That is, $r = \frac{1}{2} R\sqrt{3}$.

Since $\angle\ ACB$ is measured by arc AB, $\frac{1}{6}$ of the circumference, it equals 60°. And, since $\angle\ A$ is measured by $\frac{1}{2}$ of $\frac{2}{3}$ of the circumference, it is equal to 120°.

4. In a regular inscribed decagon $a = \dfrac{R(\sqrt{5}-1)}{2}$, $r = \frac{1}{4} R\sqrt{10 + 2\sqrt{5}}$, $A = 144°$, $C = 36°$.

Let a be the side of a regular inscribed decagon, C the centre of the $\odot$, R the radius, r the $\perp$ drawn

from C to the side a. Since a, the side of the decagon, is equal to the greater segment of the radius when divided in extreme and mean ratio (§ 394), $R : a :: a : R - a$. $\therefore a^2 = R^2 - aR$. $\therefore a^2 + aR = R^2$. $\therefore a^2 + (\quad) + \frac{1}{4} R^2 = \frac{5 R^2}{4}$. $\therefore a + \frac{1}{2} R = \frac{R}{2} \sqrt{5}$. $\therefore a = \frac{R(\sqrt{5} - 1)}{2}$.

Again, $R^2 = r^2 + (\frac{1}{2} a)^2$. But $R^2 = a^2 + aR$; $\therefore a^2 + aR = r^2 + \frac{1}{4} a^2$. Substitute the value of a; then $\frac{R^2 (5 - 2\sqrt{5} + 1)}{4} + \frac{R^2 (\sqrt{5} - 1)}{2} = r^2 + \frac{R^2 (5 - 2\sqrt{5} + 1)}{16}$.

$$\therefore 16 r^2 = 10 R^2 + 2 R^2 \sqrt{5}.$$

$$\therefore r^2 = \frac{R^2 (10 + 2\sqrt{5})}{16}.$$

$$\therefore r = \frac{1}{4} R \sqrt{10 + 2\sqrt{5}}.$$

Again, since all the $\measuredangle$s of a polygon $= 2(n - 2)$ rt. $\measuredangle$s, the $\measuredangle$s of the regular decagon $= 16$ rt. $\measuredangle$s. $\therefore$ each $\angle = \frac{16}{10}$ rt. $\measuredangle$s $= 144°$.

Since the radii of a regular decagon form 10 equal $\measuredangle$s at the centre, each angle at the centre $= \frac{1}{10}$ of $360° = 36°$.

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Ex. 2. Show that the area of an equilateral triangle, each side of which is denoted by a, is equal to $\frac{a^2\sqrt{3}}{4}$.

Since the area of a $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, in an equilateral Δ, each side of which is denoted by a, $s = \frac{3a}{2}$, $s - a = \frac{a}{2}$, $s - b = \frac{a}{2}$, $s - c = \frac{a}{2}$. $\therefore$ the area of the equilateral $\Delta = \sqrt{\frac{3a^4}{16}} = \frac{a^2\sqrt{3}}{4}$.

Ex. 3. How many acres are contained in a triangle whose sides are respectively 60, 70, and 80 chains.

Ans. 203.331 acres.

Ex. 4. How many square feet are contained in a triangle each side of which is 75 feet?

Ans. 2435.6 sq. ft.

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Ex. 1. If a plane be passed through one of the diagonals of a parallelogram, the perpendiculars to the plane from the extremities of the other diagonal are equal.

Let $ABDC$ be a ▱, AD and BC its diagonals. Through AD pass a plane, and from B and C drop ⊥s to the plane meeting the plane at P and O respectively. Join DP and AO. In the rt. △s BDP and CAO, $BD = CA$ (being opposite sides of a ▱), BD is ∥ to CA and BP is ∥ to CO (§ 458). ∴ $\angle\ DBP = \angle\ ACO$ (§ 462). ∴ $\triangle\ BDP = \triangle\ CAO$ (§ 110). ∴ $BP = CO$..

Ex. 2. If each of the projections of a line AB upon two intersecting planes be a straight line, the line AB is a straight line.

For the two projecting planes intersect in the line AB. ∴ AB is a straight line (§ 446).

Ex. 3. The height of a room is eight feet, how can a point in the floor directly under a certain point in the ceiling be determined with a ten-foot pole?

Keeping one end of the pole on the given point in the ceiling, trace a ⊙ with the other end of the pole. Then the centre of the ⊙ will be the point in the floor directly under the point in the ceiling.

Ex. 4. If a line be drawn at an inclination of $45°$ to a plane, what is the greatest angle which any line of the plane, drawn through the point in which the inclined line pierces the plane, makes with the line?

Let MN be a plane, and AB be a line drawn to the plane at an inclination of 45°. Let AC be the projection of line AB. Prolong CA to D. $\angle BAC$ is the least $\angle$ which BA makes with any line of plane MN (§ 478). $\therefore \angle BAD$ (the supplement of $\angle BAC$) is the greatest $\angle$ which the line BA makes with any line of the plane. That is, 135° is the greatest $\angle$ which the line makes with the plane.

Ex. 5. Through a given point pass a plane parallel to a given plane.

Let MN be the given plane, and O the given point. From O draw $OC \perp$ to the plane MN; and through O draw OD and $OE \perp$ to OC. Then will the plane PQ, embracing the lines OD and OE, be $\perp$ to the line OC, and $\therefore \parallel$ to MN (§ 461).

Ex. 6. Find the locus of points in space which are equally distant from two given points.

Let A and B be two given points. Join AB. At C, the middle point of AB, erect a $\perp$. Every point in this $\perp$ is at equal distances from A and B (§ 59). Hence if this $\perp$ be revolved about C, it will generate a plane every point of which will be at equal distances from A and B.

Ex. 7. Show that the three planes, passed through the edges of a trihedral angle and the bisectors of

the opposite face angles respectively, intersect in the same straight line.

Let *A–BCD* be a trihedral ∠; *AG*, *AF*, and *AE* be the bisectors of the face ∠s *BAD*, *BAC*, and *CAD* respectively. Pass the planes *BEA*, *DFA*, and *GCA*. The plane *BEA* is the bisector of the dihedral ∠ whose edge is *AB*. ∴ line *BE* bisects the face ∠ *CBD*. In like manner line *DF* bisects face ∠ *BDC*, and *CG* bisects face ∠ *BCD*. The three bisectors *DF*, *BE*, and *CG*, meet in a point *H* (§ 119). Then line *AH* is the line of intersection of the three planes.

Ex. 8. Find the locus of the points which are equally distant from the three edges of a trihedral angle.

Let *A–BCD* be a given trihedral ∠. Bisect the face ∠s *CAB*, *CAD*, and *DAB*. Through these bisectors *AA'*, *AB'*, *AC'* pass planes ⊥ to the faces. Then every point in the plane passed through *AA'* is at equal distances from the edges *AB* and *AC* (§ 477); and every point in the plane passed through *AB'* is at equal distances from the edges *AC* and *AD*; and every point in the plane passed through *AC* is at equal distances from the edges *AB* and *AD*. ∴ their intersection *AH* is at equal distances from *AB*, *AC*, and *AD*.

Ex. 9. Cut a given quadrahedral angle by a plane so that the section shall be a parallelogram.

Let A–$BCDE$ be a quadrahedral $\angle$. Through the opposite edges AB, AD, and AE, AC pass planes intersecting in AM. Through O, any point in AM, draw in the plane ACE a straight line cutting AC in K and AE in G, and such that $OK = OG$. Likewise through O in the plane ABD draw a straight line cutting AB in F and AD in H, and such that $OF = OH$. Then the section $FGHK$ made by a plane passed through KG and FH is a $\square$ (converse of § 148).

Ex. 10. Determine a point in a given plane such that the sum of its distances from two given points on the same side of the plane shall be a minimum.

Let A and B be the two given points. From A draw a $\perp$ to the given plane meeting it at C. Prolong AC to D making $CD = AC$. Join DB intersecting the given plane at E. Then E is the point required. Let F be any other point in the given plane. Join AE, BF, and AF. Now $AE = DE$, and $AF = DF$ (Ex. 6). And $DF + FB > BD$. $\therefore AF + FB > BD$. That is, $AF + FB > DE + EB$. $\therefore AE + EB < AF + FB$.

Ex. 11. Determine a point in a given plane such that the difference of its distances from two

given points on opposite sides of a plane shall be a maximum.

Let A and B be any two points on opposite sides of the given plane. Suppose B to be nearer to the plane than A. Draw $BE \perp$ to the plane. Prolong BE to D, making $ED = BE$. Join AD. Prolong AD until it meets the plane at C. Then C will be the point required. Join CB, and let H be any other point in the plane. Join DH and HB. Now $CD = CB$, and $HD = HB$ (Ex. 6). $AC - CB = AC - CD = AD$. But $AD > AH - HD$ (§ 97). $\therefore AD > AH - HB$. $\therefore AC - CB > AH - HB$.

PAGE 311.

Ex. 1. Given a cubical tank holding one ton of water; find its length in feet, if a cubic foot of water weigh 1000 ounces.

Ans. 3.17 + ft.

Ex. 2. At 17 cents a square foot, what is the cost of lining with zinc a rectangular cistern 5 feet 7 inches long, 3 feet 11 inches broad, 2 feet 8½ inches deep?

Ans. \$12.46+.

Ex. 3. Find the side of a cubical block of cast iron weighing a ton, if iron weigh 7.2 as much

as water, and a cubic foot of water weigh 1000 ounces.

Ans. 1.64+ ft.

Ex. 4. How many cubic yards of gravel will be required for a walk surrounding a rectangular lawn 200 yards long, and 100 yards wide; the walk to be 3 feet wide, and the gravel 3 inches deep?

Ans. $50\frac{1}{3}$ cu. yds.

Ex. 5. The volume of a rectangular solid is the sum of two cubes whose edges are 10 inches and 2 inches respectively, and the area of its base is the difference between two squares whose sides are $1\frac{1}{9}$ feet and $1\frac{2}{9}$ feet respectively; find its altitude in feet.

Ans. 2.25 ft.

Ex. 6. A rectangular cistern whose length is equal to its breadth is 22 decimetres deep, and contains 10 tonneaux of water; find its length.

Ans. 21.32+ dm.

Ex. 7. Given a regular prism whose base is a regular hexagon inscribed in a circle 6 meters in diameter, and whose altitude is 8.7 metres; find the number of kilolitres it will contain, if the thickness of the walls be 1 decimeter.

Ans. 381.834 kl.

Ex. 8. A pond whose area is 11 hectares, 21 ares, 22.2 centares, is covered with ice 21 centimetres thick. What is the weight of this body of ice in kilogrammes, the weight of ice being 92 per ct. that of water.

Ans. 21,662,009+ kgs.

Ex. 9. Given two hollow oblique prisms whose interior dimensions are as follows: the area of a right section of the first is 18 square feet, of the second 2.1 square metres; a lateral edge of the first is 9 feet, of the second 2.1 metres; find the volume of each in cubic yards, cubic metres, cubic decimetres, and cubic centimetres; find the capacity of each in gallons and litres, in bushels and hectolitres; and find the weight of water in pounds and in kilogrammes required to fill each prism.

Ans. FIRST PRISM.	SECOND PRISM.
6 cu. yds.	5.768+ yds.
4.587+ cu. m.	4 cu. m.
4587+ cu. dm.	4410 cu. dm.
4,587,155 cu. cm.	4,410,000 cu. cm.
1211.84 gals.	1165 gals.
4587 l.	4410 l.
130+ bu.	125+ bu.
45.87 hl.	44.1 hl.
10,112 lbs.	9722 lbs.
4587 kg.	4410 kg.

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Ex. 1. Given a pyramid whose base is a rectangle 80 feet by 60 feet, and whose lateral edges are each 130 feet; find its volume, and its entire surface.

Ans. Surface, 22,280 sq. ft.; volume, 192,000 cu. ft.

Ex. 2. Given the frustum of a pyramid whose bases are heptagons; each side of the lower base being 10 feet, and of the upper base 6 feet, and the slant height 42 feet; find the convex surface in square yards.

Ans. $261\frac{1}{3}$ sq. yds.

Ex. 3. Given a stick of timber 30 feet long, the greater end being 18 inches square, and the smaller end 15 inches square; find its volume in cubic feet.

Ans. 56.8+.

Ex. 4. Given a stone obelisk in the form of a regular quadrangular pyramid, having a side of its base equal to 25 decimetres, and its slant height 12 metres. The stone weighs 2.5 as much as water. What is its weight in kilogrammes?

Ans. 62,156+ kg.

Ex. 5. Given the frustum of a pyramid whose bases are squares; each side of the lower base being 35 decimetres, each side of the upper base 25 decimetres, and the altitude 15 metres; find its volume in steres.

Ans. 136.25 st.

Ex. 6. Given a right hexagonal pyramid whose base is inscribed in a circle 30 feet in diameter, and whose altitude is 20 feet; find its convex surface, and its volume.

Ans. Convex surface, 1073.25 sq. ft.; volume, 3897.2 cu. ft.

Ex. 7. Given a right pentagonal pyramid whose base is inscribed in a circle 20 feet in diameter, and whose slant height is 30 feet; find its convex surface, and its volume.

Ans. Convex surface, 881.6+ sq. ft.; volume, 2289 cu. ft.

Ex. 8. Find the difference between the volume of the frustum of a pyramid, and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases, and equidistant from them.

Let B and b be the lower and upper bases of the frustum respectively; B' be the base of the

prism, and H the common altitude. Let V and V' denote the volumes of the frustum and of the prism respectively. Then,

$$V' - V = \frac{H}{3}(3B' - B - b - \sqrt{Bb}).$$

Ex. 9. Given a stick of timber 32 feet long, 18 inches wide, 15 inches thick at one end, and 12 inches at the other; find the number of cubic feet, and the number of feet *board measure* it contains. Find equivalents for the results in the metric system.

Ans. 47.8885+ cu. ft.; 574.662+ ft., board measure; 1.35523+ cu. m.; 533.859+ sq. dm.

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Ex. 1. The portion of a tetrahedron cut off by a plane parallel to any face is a tetrahedron similar to the given tetrahedron.

The faces of the portion cut off are equal in number and similar to the corresponding faces of the tetrahedron. The section made by the cutting plane is a $\triangle$ similar to the parallel face of the tetrahedron (§ 565). $\therefore$ trihedral $\measuredangle$s of the portion cut off will be equal to the corresponding trihedral $\measuredangle$s of the tetrahedron (§ 492). $\therefore$ portion cut off is a tetrahedron similar to the entire tetrahedron.

Ex. 2. Two tetrahedrons, having a dihedral angle of one equal to a dihedral angle of the other, and the faces including these angles respectively similar, and similarly placed, are similar.

Let $A\text{–}BCD$ and $A'\text{–}B'C'D'$ be the tetrahedrons, and let the dihedral $\angle$ made by the faces BAC and $DAC =$ the dihedral $\angle$ made by the faces $B'A'C'$ and $D'A'C'$.

We can place $A'\text{–}B'C'D'$ so that $A'C'$ shall be $\parallel$ to AC, and the face $A'B'C'$ $\parallel$ to the face ABC. Then, since the dihedral $\angle$s are equal, the face $A'C'D'$ will be $\parallel$ to the face ACD. And since $A'B'C'$ is similar to ABC, $\therefore$ $A'B'$ is $\parallel$ to AB, and $A'D'$ is $\parallel$ to AD. $\therefore \angle B'A'D' = \angle BAD$ (§ 462). From this it follows that the trihedral $\angle$s at A and A', being composed of plane $\angle$s which are equal each to each, are equal.

In the same way it can be proved that the trihedral $\angle$s at C and C' are equal.

Again, the faces ABD and $A'B'D'$ are $\parallel$ planes, and the faces CBD and $C'B'D'$ are $\parallel$ planes (§ 463); $\therefore$ BD, the intersection of the planes ABD and CBD, is $\parallel$ to $B'D'$, the intersection of the planes $A'B'D'$ and $C'B'D'$. $\therefore \angle DBA = \angle D'B'A'$, and $\angle CBD = \angle C'B'D'$. And $\angle ABC = \angle A'B'C'$ (hyp.). $\therefore$ trihedral $\angle$s at B and B' are equal. In like manner it follows that the trihedral $\angle$s at D and D' are equal. $\therefore$ the tetrahedrons are similar.

Ex. 3. Given two similar polyhedrons, whose volumes are 125 feet and 12.5 feet respectively; find the ratio of two homologous edges.

Ans. 2.1544+ : 1.

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Ex. 1. Required, the entire surface and volume of a cylinder of revolution whose altitude is 30 inches, and whose base is a circle of which the diameter is 20 inches.

Ans. Surface $= 800\pi$; volume $= 3000\pi$.

Ex. 2. Required, the volume of a right truncated triangular prism the area of whose base is 40 square inches, and whose lateral edges are 10, 12, and 15 inches, respectively.

Ans. $493\frac{1}{3}$ cu. in.

Ex. 3. Let E denote an edge of a regular tetrahedron; show that the altitude of the tetrahedron is equal to $E\sqrt{\frac{2}{3}}$; that the surface is equal to $E^2\sqrt{3}$; and that the volume is equal to $\frac{E^3}{12}\sqrt{2}$.

Let H denote the altitude of the tetrahedron. The altitude will meet the base (which is an equilateral $\triangle$) at the centre of the $\odot$ circumscribed about the base. Let R equal the radius of this $\odot$. Then E, H, and R are the sides of a rt. $\triangle$, E being the hypotenuse. $\therefore H^2 = E^2 - R^2$.

To find R in terms of E, inscribe in the ⊙ above mentioned a hexagon, by bisecting the three arcs which correspond to the sides of the base. Now, each side of the base $= E$, and each side of the hexagon $= R$ (§ 391). And one side E of the base, one side R of the hexagon, and a diameter $2R$ of the ⊙, form a rt. △ (§ 204).

$\therefore (2R)^2 = E^2 + R^2$, whence $R^2 = \dfrac{E^2}{3}$. $\therefore H^2 = E^2 - \dfrac{E^2}{3} = \dfrac{2E^2}{3}$. $\therefore H = E\sqrt{\tfrac{2}{3}}$.

The area of one face $= E \times \frac{1}{2}$ the altitude of the face. This altitude, $\dfrac{E}{2}$, and E, form the sides of a rt. △, E being the hypotenuse. $\therefore$ the altitude $= \dfrac{E}{2}\sqrt{3}$. $\therefore$ the area of one face $= \dfrac{E^2}{4}\sqrt{3}$. $\therefore$ the entire surface of the tetrahedron $= \dfrac{E^2}{4}\sqrt{3} \times 4 = E^2\sqrt{3}$.

The volume of the tetrahedron $= \dfrac{\text{base} \times \text{altitude}}{3}$

$$= \frac{\dfrac{E^2}{4}\sqrt{3} \times E\sqrt{\tfrac{2}{3}}}{3} = \frac{E^3}{12}\sqrt{2}.$$

Ex. 4. Required, the number of quarts that a cylinder of revolution will contain whose height is 20 inches, and the diameter of whose base is 12 inches.

Ans. 39+.

Ex. 5. Given S, the surface of a cube, find its edge, diagonal, and volume. What do these become when $S = 54$?

Ans. Edge $= \sqrt{\frac{S}{6}}$; diagonal $= \sqrt{\frac{S}{2}}$; volume $= \sqrt{\frac{S^3}{216}}$. When $S = 54$ these become 3; $3\sqrt{3}$; 729.

Page 364.

Ex. 1. Given a cone of revolution whose side is 24 feet, and the diameter of its base 6 feet; find its entire surface, and its volume.

Ans. $T = \pi R\,(L + R)$ (§ 659) $= 254.4696$ sq. ft.; $V = \frac{1}{3}\,\pi R^2 \times H$ (§ 665) $= 224.4$ cu ft.

Ex. 2. Given the frustum of a cone whose altitude is 24 feet, the circumference of its lower base 20 feet, and that of its upper base 16 feet; find its volume.

Ans. $V = \frac{1}{3}\,\pi H\,(R^2 + r^2 + Rr) = 1618.5$ cu. ft.

Ex. 3. The volume of the frustum of a cone of revolution is 8025 cubic inches; its altitude 14 inches; the circumference of the lower base twice that of the upper base. What are the circumferences of the bases?

Ans. $V = \frac{1}{3}\,\pi H\,(R^2 + r^2 + Rr)$. $\therefore 8025 = \frac{1}{3} \times 3.1416 \times 14\,(2\,r^2) + r^2 + 2\,r^2$. Circumference of upper base $= 55.3+$ in.; circumference of lower base $= 110.6$ in.

Ex. 4. The frustum of a cone of revolution whose altitude is 20 feet, and the diameters of its bases 12 feet and 8 feet respectively, is divided into two equal parts by a plane parallel to its bases. What is the altitude of each part?

Ans. 11.9+ ; 8.1−.

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Ex. 1. The surface of a cone is 540 square inches; what is the surface of a similar cone whose volume is 8 times as great?

Ans. 2160 sq. in.

Ex. 2. The lateral surface of a cone is S; what is the lateral surface of a similar cone whose volume is n times as great?

Ans. $S\sqrt[3]{n^2}$.

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Ex. 1. Show that two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each.

Let P and P' be homologous vertices of two symmetrical polyhedrons. On any two corresponding faces not adjacent to P and P', and from two homologous vertices of these faces, draw diagonals dividing these faces into △. From P and P' draw straight lines to the vertices of these △. Repeat this con-

struction for each of the faces not adjacent to P and P'. Then the polyhedrons will be divided into the same number of tetrahedrons; that is, into as many tetrahedrons as there are $\triangle$s in these faces. Now any two corresponding tetrahedrons, as $P\text{–}ABC$ and $P'\text{–}A'B'C'$, are symmetrical. For, in the $\triangle$s ABC and $A'B'C'$, $AB = A'B'$, $BC = B'C'$, $\angle ABC = \angle A'B'C'$. $\therefore \triangle ABC = \triangle A'B'C'$. In the $\triangle$s ABP and $A'B'P'$, $AB = A'B'$, $BP = B'P'$, $\angle ABP = \angle A'B'P'$. In like manner $\triangle CBP = \triangle C'B'P'$, and $\triangle ACP = \triangle A'C'P'$. $\therefore$ the faces of these tetrahedrons are equal each to each, but have their parts arranged in reverse order. $\therefore$ the two tetrahedrons, $P\text{–}ABC$ and $P'\text{–}A'B'C'$, are symmetrical. In like manner any other two corresponding tetrahedrons may be shown to be symmetrical.

Ex. 2. Show that two symmetrical polyhedrons are equivalent.

Since two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each, it is only necessary to prove that two symmetrical tetrahedrons are equivalent. Let two symmetrical tetrahedrons, $P\text{–}ABC$ and $P'\text{–}A'B'C'$, be so placed that the base ABC shall coincide with the base $A'B'C'$ in the plane MN, having the vertex P above the plane, and the vertex P' below the plane. Then, since P and P'

are symmetrical with respect to the plane MN, the altitudes of the two tetrahedrons PO and $P'O$ are equal. ∴ the tetrahedrons are equivalent (§ 575).

Ex. 3. Show that the intersection of two planes of symmetry of a solid is an axis of symmetry.

If the solid AB have two planes of symmetry, MN and PQ, EF, the line of intersection of these planes, will be an axis of symmetry, because it contains all the points common to the planes.

Ex. 4. Show that the intersections of three planes of symmetry of a solid are three axes of symmetry; and that the common intersection of these axes is the centre of symmetry.

Let the solid AB have three planes of symmetry, MN, PQ, and RS, ⊥ to each other. Then the intersections of these planes, EF, CD, and LK, will (by Ex. 3) be axes of symmetry. ∴ O, the point common to these lines of intersection, will be a centre of symmetry.

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Ex. 1. The volume of a cone is 1728 cubic inches; what is the volume of a similar cone whose surface is 4 times as great?

Ans. 13,824 cu. in.

Ex. 2. The volume of a cone is V; what is the volume of a similar cone whose surface is n times as great?

Ans. $Vn\sqrt{n}$.

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Ex. 1. The altitude of a cone of revolution is 12 inches; at what distances from the vertex must three planes be passed parallel to the base of the cone in order to divide the lateral surface into four equal parts?

Ans. 6 in.; 8.474+ in.; 10.39+ in.

Ex. 2. The altitude of a given solid is 2 inches, its surface 24 square inches, and its volume 8 cubic inches; find the altitude and surface of a similar solid whose volume is 512 cubic inches.

Ans. Altitude, 8 in.; surface, 384 sq. in.

Ex. 3. The volumes of two similar cones of revolution are 6 cubic inches and 48 cubic inches respectively, and the slant height of the first is 5 inches; find the slant height of the second.

Ans. 10 in.

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Ex. 1. If, from the extremities of one side of a spherical triangle, two arcs of great circles be drawn

to a point within the triangle, the sum of these arcs is less than the sum of the other two sides of the triangle.

In the spherical $\triangle$ ABC, let the arcs BD and CD be drawn to any point D within the $\triangle$. Produce BD to meet AC in E. Then $DC < DE + EC$ (§ 722); and $BD + DE < AB + AE$. Adding these inequalities, $BD + DC < AB + AC$.

Ex. 2. On the same sphere, or on equal spheres, if two spherical triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first will be greater than the third side of the second.

In the spherical $\triangle$ ABC and $A'B'C'$, let $AB = A'B'$, $AC = A'C'$, $\angle A > \angle A'$; then is $BC > B'C'$. If the parts are arranged in the same order, upon $\triangle$ ABC apply $\triangle$ $A'B'C'$, so that $A'B'$ shall coincide with AB. Since $\angle$ $A' < \angle$ A, side $A'C'$ will fall within AC. Bisect $\angle$ CAC' by arc AP meeting BC at P. Draw arc PC'. Then $\triangle ACP = \triangle AC'P$ (§ 741). $\therefore PC' = PC$. Now $BP + PC' > BC'$ (§ 722). Substitute for PC' its equal PC, then $BP + PC > BC'$; or $BC > BC'$.

If the parts are arranged in reverse order, construct a $\triangle$ symmetrical with ABC, and upon this symmetrical $\triangle$ apply $\triangle$ $A'B'C'$.

Ex. 3. To draw an arc perpendicular to a given spherical arc, from a given point without it.

Let AB be the given spherical arc, and C a point without it. From C as a pole draw an arc of a great $\odot$ intersecting the given arc (produced if necessary) at the points E and F. Find a point H at a quadrant's distance from E and from F; and draw arc CH. Then CH will be the $\perp$ required.

Ex. 4. At a given point in a given arc, to construct a spherical angle equal to a given spherical angle.

Let ACB be the given $\angle$, and let F be the given point in the given arc MN. From C as a pole describe an arc of a great $\odot$ intersecting the sides of the $\angle$ at E and F. From F as a pole describe the indefinite arc MX. On MX take $MK = EF$, and draw the arc FK. $\angle MFK$ is the $\angle$ required.

Ex. 5. To inscribe a circle in a given spherical triangle.

Let ABC be the given spherical $\triangle$. Bisect $\angle$s A and B by arcs of great ⓢ which meet at O. From O draw arc $OD \perp$ to AB. From O as a pole, with arc OD, describe a $\odot$. It will be the $\odot$ required.

Ex. 6. Given a spherical triangle whose sides are 60°, 80°, and 100°; find the angles of its polar triangle.

Ans. 120°, 100°, 80°.

Ex. 7. The volume of a pyramid is 200 cubic feet; find the volume of a similar pyramid which is three times as high.

Ans. 5400 cu. ft.

Ex. 8. Find the centre of a sphere whose surface shall pass through three given points and touch a given plane.

Let P_1, P_2, P_3, denote the given points, E the given plane. Construct a ⊙ whose circumference shall pass through P_1, P_2, P_3, and at its centre K erect a ⊥ to the plane of the ⊙; then the centre of the sphere must lie in this ⊥. From K let fall a ⊥ to the plane E; then this ⊥, together with the former one, determines a plane which is ⊥ to the plane E and to the plane of the ⊙, cutting the former in a line c, and the latter in a diameter d of the ⊙. In this plane construct a ⊙ whose circumference shall pass through the extremities of the diameter d, and also touch the line c; then the centre of this ⊙ will be the centre of the sphere required.

The problem admits of two solutions.

Ex. 9. Find the centre of a sphere whose surface shall pass through three given points, and shall also touch the surface of a given sphere.

Let P_1, P_2, P_3, be the given points, K the centre of the given sphere. The centre of the sphere

required must lie in the $\perp$ erected at the centre of the $\odot$ whose circumference passes through P_1, P_2, and P_3. Imagine a plane E passed through this $\perp$ and K; it will cut the $\odot$ in a diameter d, and the given sphere in a great $\odot$ G. If we now construct a $\odot$ whose circumference shall pass through the extremities of d, and shall also touch G, then the centre of this $\odot$ will also be the centre of the sphere required.

Two solutions are possible.

Ex. 10. Find the centre of a sphere whose surface shall touch two given planes, and also pass through two given points lying between the planes.

Let E_1, E_2, denote the given planes, P_1, P_2, the given points. If we bisect the dihedral angle between E_1 and E_2 by a plane E, the centre of the required sphere must lie in this bisecting plane. Let fall $P_1M \perp$ to E, and let it intersect E in M; produce P_1M making $MP = P_1M$, and construct (by Ex. 8) a sphere whose surface shall pass through P_1, P_2, and P, and shall also touch either of the planes E_1, E_2. This sphere is the sphere required.

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Ex. 1. Given the radius of a sphere is 10 feet; find the area of a lune whose angle is 30°.

Ans. 104.72 sq. ft.

Ex. 2. Given the diameter of a sphere is 16 feet; find the area of a lune whose angle is 75°.

Ans. 167.55 sq. ft.

Ex. 3. Given the diameter of a sphere is 20 inches; find the entire surface of its circumscribed cylinder; and of its circumscribed cone, the vertical angle of the cone being 60°.

Ans. Surface of cylinder = 1884.95 sq. in.; surface of cone = 2827.44 sq. in.

PAGE 397.

Ex. 1. Given a sphere whose diameter is 20 inches; find the circumference of a small circle whose plane cuts the diameter 4 inches from the centre.

Ans. 57.55 in.

Ex. 2. Construct on the spherical blackboard spherical angles of 30°, 45°, 90°, 120°, 150°, and 135°.[1]

From any point *P* as a pole, with a quadrant arc of 90°, describe the circumference of a great ⊙ *ABCD*. Draw *PB*, an arc of a great ⊙. From

[1] For measuring arcs on the spherical blackboard, a strip of paper equal in length to the circumference of a great circle of the given sphere, and marked off in degrees, will be found convenient.

B lay off on the circumference $ABCD$ arcs of 30°, 45°, etc. Pass arcs of great Ⓢ through P, and the extremities, MN, etc., of these arcs. Then ∠s BPM, BPN, etc., will be spherical ∠s of 30°, 45°, etc.

Ex. 3. Construct on the spherical blackboard a spherical triangle whose sides are 100°, 80°, and 70° respectively. What is true of its polar triangle?

From A, a point on the circumference of a great ⊙, lay off arc AB on this circumference = 100°. From A and B as poles, with arcs = 80° and 70° respectively, describe arcs intersecting at C. Through A,C and B,C draw arcs of great Ⓢ. △ ABC is the △ required.

The ∠s of the polar △ are respectively 80°, 100°, and 110°.

Ex. 4. Find the surface and volume of a sphere whose radius is 10 inches; also find the area of a spherical triangle on this sphere, the angles of the triangle being 80°, 85°, and 100° respectively.

Ans. Surface of sphere = 1256.64 sq. in.; volume of sphere = 4188.8 cu. in. Area of △ = 148.35 sq. in.

Ex. 5. If 7 equidistant planes cut a sphere, each perpendicular to the same diameter, what are the relative areas of the zones?

Ans. They are equal.

Ex. 6. Given two mutually equiangular triangles on spheres whose radii are 10 inches and 40 inches respectively; what are their relative areas?

Ans. The ratio 1 : 16.

Ex. 7. Let V denote the volume of a spherical pyramid, S its base, E the spherical excess of its base, and R the radius of the sphere; show that $S = \frac{1}{2}\pi R^2E$, and $V = \frac{1}{6}R^3E$.

Ans. Surface of sphere $= 4\pi R^2$; surface of trirectangular $\triangle = \frac{1}{2}\pi R^2$. $\therefore S = \frac{1}{2}\pi R^2E$ (§ 771).

Since base of pyramid $= \frac{1}{2}\pi R^2E$, volume of pyramid $= \frac{1}{2}\pi R^2E \times \frac{1}{3}R = \frac{1}{6}\pi R^3E$.

Ex. 8. Given the volume of a sphere 1728 inches; find its radius.

Ans. 7.4 +.

Ex. 9. Find the ratio of the surfaces, and the ratio of the volumes, of a cube and of the inscribed sphere.

Ans. $6 : \pi$.

Ex. 10. Find the ratio of the surfaces, and the ratio of the volumes, of a sphere and the circumscribed cylinder.

Ans. 2 : 3.

Ex. 11. Let V denote the volume and H the altitude of the spherical segment of one base, and R the radius of the sphere; show that $V = \pi H^2 (R - \frac{1}{3} H)$. Also, find V when $R = 12$ and $H = 3$.

Let V' = volume of sector, V'' = volume of cone; then $V = V' - V''$. $V'' = \frac{\pi R''^2 \times H''}{3}$.

Now $H'' = R - H$, and $R''^2 = R^2 - (R - H)^2 = 2\,RH - H^2$. $\therefore V'' = \frac{2\,\pi RH - \pi H^2}{3}(R - H)$

$= \frac{2\,\pi R^2 - 3\,\pi RH^2 + \pi H^3}{3}$. But $V' = \frac{2\,\pi R^2 \times H}{3}$.

Substitute these values in the equation $V = V' - V''$;

$$V = \frac{2\,\pi R^2 \times H}{3} - \frac{2\,\pi R^2 \times H - 3\,\pi R \times H^2 + \pi H^3}{3}$$

$$= \frac{3\,\pi RH^2 - \pi H^3}{3} = \pi H^2 (R - \tfrac{1}{3} H).\quad 311, \text{nearly}.$$

Ex. 12. Given a sphere 2 feet in diameter; find the volume of a segment of the sphere included between two parallel planes, one at 3 and the other at 9 inches from the centre. (Two solutions.)

Ans. 1979.2 cu. in.; 4637 cu. in.

Ex. 13. A sphere 4 inches in diameter is bored through the centre with a two-inch augur; find the volume remaining.

Ans. 21.76 cu. in.

SELECTIONS

FROM

COLENSO'S GEOMETRICAL EXERCISES

AND

KEY.

EXERCISES.

I. — Rectilinear Figures.

Ex. 1. From two given points on the same side of a given line, draw two lines which shall meet in that line and make equal angles with it.

Given A, B; draw $AC \perp$ to given line, and produce, making $CD = AC$; join BD, cutting the given line CEF at E; then in $\triangle$s AEC, DEC, we have $\angle AEC = \angle DEC = \angle BEF$.

Ex. 2. Describe a square of which a given line shall be the diagonal.

Bisect given line AB at rt. $\angle$s by DCE, and make $CD = CE = CA$ or CB; then $AD = BD =$ &c., and the figure is equilateral: also $\angle CAD = \angle CDA = \frac{1}{2}$ rt. $\angle$, and, similarly, $\angle CDB = \frac{1}{2}$ rt. $\angle$; $\therefore \angle ADB =$ rt. $\angle$, &c., and the figure is also rectangular.

Ex. 3. Let AB bisect CD at right angles at B; from any point E draw EC through the nearest extremity of CD cutting AB at F; and prove that the difference of EF and DF is greater than that of any other two lines drawn from E and D to meet in the line AB.

For, if G be any other point in AB, $EG \pm DG = EG \pm CG$ which is $< EC$, *i.e.* $< EF \pm DF$.

Ex. 4. The sum of the diagonals of a quadrilateral is less than the sum of any four lines that can be drawn from any point whatever (except the intersection of the diagonals) to the four angles.

AC, BD the diagonals, E any point; then $AC < AE + EC$, and $BD < BE + ED$, $\therefore$ &c.

Ex. 5. Construct a triangle, having given one side, an adjacent angle, and the sum or difference of the other two sides.

Given side AB, and $\angle ABC$; in BC take $BD =$ given sum or difference, in the former case $>$, in the latter $<$, AB, and at A in DA make $\angle DAC = \angle ADC$; then $AC = CD$, and ACB is the $\triangle$.

Ex. 6. From a given point draw three lines of given lengths, so that their extremities may be in one line, and equally distant from each other.

Given A: draw any line AB, and make $AC = CD =$ one of given lines; construct $\triangle AED$, with sides AE, $DE =$ other given lines; in EC produced make $CF = CE$: then AF, AC, AE are the lines required, since $AF = DE$.

Ex. 7. If a line which bisects the vertical angle of a triangle also bisects the base, the triangle is isosceles.

Produce AD (the bisecting line), making $DE = DA$; then in $\triangle$s ADB, EDC, $\angle BAD = \angle CED$, and similarly, $\angle CAD = \angle BFD$; $\therefore$ in $\triangle$s BAE, CAE, $BA = AC$.

Ex. 8. From a given point draw a line making equal angles with two given lines.

Given A, BC, BD; bisect $\angle CBD$ by BE, and through A draw $CED \perp$ to BE; then in $\triangle$s BEC, BED, $\angle BCE = \angle BDE$.

Ex. 9. Through a given point draw a line, so that the perpendiculars upon it from two other given points may be equal to each other.

Given A, and B, C; bisect BC at D, and draw BE, CF $\perp$ on AD; then $BE = CF$.

Ex. 10. Given the perimeter of a triangle, and the angles at the base: to construct the triangle.

Given $AB =$ the perimeter, BAC, $ABD =$ the $\angle$s at the base; bisect these $\angle$s by AE, BE; from E to AB draw EF, $EG \parallel$ to AC, BD; then EFG shall be the triangle required. For $\angle EAF = \angle EAC = \angle AEF$, $\therefore AF = FE$; similarly, $EG = GB$; $\therefore$ perimeter $EFG = AB$; and $\angle$s EFG, $EGF =$ $\angle$s BAC, ABD.

Ex. 11. The parallelogram whose diagonals are equal is rectangular.

The diagonals AC, BD are $=$, and bisect each other at E; $\therefore EA = EB$, and $\angle EAB = \angle EBA$; so $\angle EAD = \angle EDA$; $\therefore \angle BAD =$ $\angle$s $ABD + ADB =$ rt. $\angle$.

Ex. 12. The quadrilateral figure whose diagonals bisect each other is a parallelogram.

In $\triangle$s AEB, CED, $\angle EAB = \angle ECD$, $\therefore AB \parallel$ to CD, &c.

Ex. 13. Draw a line which would, if produced, bisect the angle between two given lines, without producing them to meet.

Through any point P draw PC, $\parallel$ to AB, one given line, to meet CD, the other given line; bisect $\angle PCD$ by CE, cutting AB at E; then $\angle BEC = \angle ECP = \angle ECD$; and line bisecting EC at rt. $\angle$s is the line required.

Ex. 14. Draw a line DE, parallel to the base BC, and intercepted by the sides, of a triangle ABC, so that DE shall be equal to the difference of BD and CE.

Bisect $\angle B$ by BG, and exterior $\angle ACF$ by CG; and draw $DEG \parallel$ to BC: then $\angle DGB = \angle GBC = \angle DBG$; $\therefore$ $BD = DG$; so $CE = EG$: $\therefore DE = BD \pm CE$.

Ex. 15. If AB be bisected at C, and from A, C, B parallel lines be drawn cutting a given line at D, F, E, show that, according as A and B lie on the *same* or *opposite* sides of the line, CF is equal to the semi-sum or semi-difference of AD and BE.

Draw $GCH \parallel$ to given line, cutting AD, BE (or these produced) at G, H; then $AG = BH$, and $CF = \frac{1}{2}(GD + HE)$, hence, —

(*i.*) when A, B, are on the same side of the line (A nearest to the line)

$$CF = \tfrac{1}{2}[GA + AD + BE - BH] = \tfrac{1}{2}[AD + BE];$$

(*ii.*) when A, B, are on opposite sides of the line (A, C on the same side of it),

$$CF = \tfrac{1}{2}[AD - AG + HB - BE] = \tfrac{1}{2}[AD - BE].$$

Ex. 16. Trisect a right angle.

Let BAC be rt. $\angle$: on AB describe equilateral $\triangle ADB$, and bisect $\angle BAD$ by AE: then $\angle BAD = \frac{2}{3}$ rt. $\angle$, $\therefore \angle BAE = \frac{1}{3}$ rt. $\angle = \angle EAD = \angle DAC$.

Ex. 17. One of the acute angles of a right-angled triangle is three times as great as the other: trisect the smaller of these.

Since the smaller $\angle ABC = \frac{1}{4}$ rt. $\angle$, its third $= \frac{1}{12}$ rt. $\angle$. Trisect the rt. $\angle BAC$; and bisect its third part, $\angle BAD$, by AE, and again bisect $\angle BAE$ by AF; at B in AB make

$\angle ABF = \angle BAF = \frac{1}{12}$ rt. $\angle$; then bisecting $\angle CBF$, the $\angle ABC$ is trisected.

Ex. 18. If the base of an isosceles triangle be produced, twice the exterior angle is greater than two right angles by the vertical angle.

Exterior $\angle ACD = \angle s\ BAC + ABC = \angle BAC + \frac{1}{2}$ (2 rt. $\angle s - \angle BAC) = \frac{1}{2} \angle BAC +$ rt. $\angle$; $\therefore 2 \angle ACD = \angle BAC$ + 2 rt. $\angle s$.

Ex. 19. If there be an isosceles and an equilateral triangle upon the same base, and so that the inner vertex is equally distant from the other and from the extremities of the base, then, according as that of the former is the inner or outer vertex, its base angle will be $\frac{1}{4}$ or $2\frac{1}{2}$ times the vertical.

BAC the equilateral, BDC, $BD'C$, the isosceles $\triangle s$: draw $D'ADE \perp$ to BC: then $\angle BDE = 2 \angle BAD = \frac{2}{3}$ rt. $\angle$; $\therefore$ $\angle BDC = \frac{4}{3}$ rt. $\angle$, and $\angle DBE = \frac{1}{3}$ rt. $\angle = \frac{1}{4} \angle BDC$. Again, $\angle BD'E = \frac{1}{2} \angle BAD = \frac{1}{6}$ rt. $\angle$; $\therefore \angle D'BE = \frac{5}{6}$ rt. $\angle = \frac{5}{2} \angle BD'C$.

Ex. 20. Trisect a given line: and hence divide an equilateral triangle into nine equal parts.

On AB describe equilateral $\triangle ACB$; bisect $\angle s\ A, B$ by AD, BD, and draw $DE, DF, \parallel$ to CA, CB; then $\angle EDA = \angle DAC = \angle DAE, \therefore ED = EA$, and so $FD = FB$; but $\angle DEF = \angle CAB$, and $\angle DFE = \angle CBA, \therefore \angle EDF = \angle C$, and $\triangle DEF$ is equilateral: hence $EF = ED = AE$; similarly $EF = FB$.

Hence joining CD, producing ED, FD, to G, H, and drawing $LDK \parallel$ to AB, the $\triangle$ is divided into nine equal $\triangle s$.

Ex. 21. In the base BC of an isosceles triangle take any point D; in CA make CE equal to CD, and let ED cut AB at F: then show that three times the angle AEF is greater than four right angles by the angle AFE.

Rt. $\angle AEF = \angle s\ ECD + EDC = \angle ECD + \frac{1}{2}$ (2 rt. $\angle s - \angle ECD) =$ rt. $\angle + \frac{1}{2} \angle ACB$; but $\angle s\ AEF + AFE = 2$ rt. $\angle s - \angle BAC = 2 \angle ACB = 4 \angle AEF - 4$ rt. $\angle s$; $\therefore 3 \angle AEF = \angle AFE + 4$ rt. $\angle s$.

Ex. 22. If the base angles of an isosceles triangle be one fourth of the vertical angle, and from it a line be drawn perpendicular to the base to meet the opposite side produced, then the part produced, the perpendicular, and the remaining side, will form an equilateral triangle.

Let $BAC = 4 \angle ABC$, $\therefore \angle ABC = \frac{1}{3}$ rt. $\angle = \angle ACB$: hence $\angle ABD = \frac{2}{3}$ rt. $\angle$, and $\angle BDC = \frac{2}{3}$ rt. $\angle$, $\therefore \angle DAB = \frac{2}{3}$ rt. $\angle$, and $\triangle ABD$ is equilateral.

Ex. 23. ABC is a triangle, right-angled at A, and having the angle B double of the angle C: show that the side CD is double of the side AB.

In BC take $BD = AB$: then since $\angle B = \frac{2}{3}$ rt. $\angle$, $\triangle ABD$ is equilateral: also $\angle DAC = \frac{1}{3}$ rt. $\angle = \angle DCA$; $\therefore DC = DA = AB$, and $CB = 2\,AB$.

Ex. 24. If the three angles of a triangle be bisected, and one of the bisecting lines be produced to meet the opposite side, the angle contained by this line produced and one of the others is equal to the angle contained by the third line and a perpendicular from their common point of intersection to the side aforesaid.

Produce AD to E; draw $DF \perp$ to BC: then $\angle s\ DAB + DBC + DCA =$ rt. $\angle = \angle s\ DBF + BDF$; $\therefore \angle BDF$

$= \angle s\ DAB + DCA = \angle s\ DAC + DCA = \angle\ CDE$, and $\therefore$ $\angle\ BDE = \angle\ CDF$.

Ex. 25. If in the sides of a square, at equal distances from the four angles, four points be taken, one in each side, the figure formed by joining them will also be a square.

Given $AE = BF = CG = DH$: then in $\triangle s\ AEH,\ GHD$, $HE = HG$, and $\angle\ AEH = \angle\ GHD$; so $EF = FG = GH = HE$, and the figure is equilateral: but exterior $\angle\ EHD = \angle s\ HAE + AEH$, and also $= \angle s\ EHG + GHD$; $\therefore \angle\ EHG = \angle\ HAE =$ rt. $\angle$, &c.: $\therefore$ the figure is a square.

Ex. 26. Let AD, AE be squares upon the sides of the right-angled triangle ABC, and drop DF, EG perpendicular on the hypotenuse BC produced; then BC and the triangle ABC are respectively equal to the sum of DF and EG, and of the triangles DBF and ECG.

Draw $AG \perp$ to BC: then since $DBA =$ rt. $\angle$, $\therefore \angle s\ DBF + ABG =$ rt. $\angle = \angle s\ ABG + BAG$, $\therefore \angle\ DBF = \angle\ BAG$, and $\triangle s\ DBF,\ BAG$ are equiangular and also equal, since $AB = BD$; similarly of $\triangle s\ ECG,\ CAG$: $\therefore BF = AG = CG$, $DF + EG = BG + CG = BC$, and $\triangle s\ DBF + ECG = \triangle\ ABC$.

Ex. 27. Given the perpendicular from the vertex on the base, and the difference between each side and the adjacent segment of the base: construct the triangle.

On opposite sides of A in the same line take AB, $AC =$ given differences; erect AD of given length $\perp$ to BC; make $\angle s\ CDE,\ BDF = \angle s\ DCE,\ DBF$: $\therefore DE = EC$, $\therefore DE - EA = CA$; so $DF - FA = BA$; and DEF is the $\triangle$ required.

Ex. 28. The lines which bisect the angles of any paral-

lelogram form a rectangular parallelogram, whose diagonals are parallel to the sides of the former.

Let AE cut DG at F, and BE cut CG at H: then exterior $\angle AEH = \triangle ABE + BAE = \frac{1}{2}$ ($\triangle CBA + BAD$) = rt. $\angle$, &c.; hence $EFGH$ is a rectangular $\square$: again, let AF cut BC at K; then in $\triangle ABE$, KBE, $AE = EK$, but in $\triangle ABE$, CDG, $AE = CG$, $\therefore EK =$ and is $\parallel$ to CG, and $\therefore EG =$ and is $\parallel$ to KC: hence EG is $\parallel$ to BC, and so also FH is $\parallel$ to AB.

Ex. 29. AD, BC are two parallel lines cut obliquely by AB and perpendicularly by AC; BED is drawn cutting AC at E, so that ED is equal to twice AB: prove that the angle DBC is one-third of the angle ABC.

Make $\angle DAF = \angle ADF$; $\therefore AF = DF$, and $\angle FAE =$ rt. $\angle - \angle ADB =$ rt. $\angle - \angle DBC = \angle AEF$: $\therefore FE = FA = FD$; $\therefore AB = AF$, and $\angle ABF = \angle AFB = 2 \angle ADF$; or $\angle ABC = 3 \angle DBC$.

Ex. 30. In a given triangle place a line which shall be terminated by the two sides, and be equal to one given line and parallel to another.

Let A be the $\angle$ nearest to second given line; from B draw $BD =$ to one line, and $\parallel$ to the other: draw DE to AC, $\parallel$ to AB, and EF to AB, $\parallel$ to BD: EF is the line required.

Ex. 31. Any straight line drawn through the bisection of the diagonal of a parallelogram to meet the sides is bisected at that point, and also bisects the parallelogram.

Through F, the bisection of BD, draw EFG to cut $ABCD$: then $EF = FG$, and $\Delta BFE = DFC$; but $\Delta BAD = \Delta BCD$, $\therefore$ figure $ADGE =$ figure $BCGE$.

Ex. 32. In any triangle ABC, if BE, CF be perpendiculars on any line through A, and D be the bisection of BC, show that $DE = DF$.

Produce ED to cut CF at G : then since $BD = DC$, and $BE \parallel$ to CG, $\therefore ED = GD$, and $\therefore DE = DF$.

Ex. 33. If from the right angle of a triangle two lines be drawn, one bisecting the base and the other perpendicular to it, they will contain an angle equal to the difference of the two acute angles of the triangle.

AD, AE the two lines : then $DA = DB = DC$, $\therefore$ exterior $\angle\, ADE\ (AB > AC) = 2\angle\, ABC$: but $\angle$s $DAE + ADE$ = rt. $\angle$ = $\angle$s $ACB + ABC$; $\therefore \angle\, DAE = \angle$s $ACB - ABC$.

Ex. 34. Find the point in the base of a triangle from which lines drawn parallel to the sides to meet them are equal.

Bisect $\angle\, BAC$ by AD, and draw DE, $DF \parallel$ to AC, AB; $\therefore DE = DF$.

Ex. 35. Through D, E, the bisections of the sides AB, AC of a triangle, draw DE, EF parallel to BE, AB; and show that the sides of the angle DCF are equal to the three lines drawn from the angles to bisect the sides.

CD is one of the three lines, and $DF = BE$ another of them : now FE, being $\parallel$ to AB, *bisects* BC at G: and $AD = BD = FE$, and also $\parallel$ to FE, $\therefore AFED$, $AFGB$ are ▱s; and $CG = BG = AF$, $\therefore AFCG$ is ▱, and $FC = AG$.

Ex. 36. Bisect a triangle by a line drawn from a given point in one of its sides.

Given D in AB, and A the $\angle$ nearest to D : bisect BC at E, draw $AF \parallel$ to DE, and join DF: $\therefore \triangle\, DEF = \triangle\, ADE$, $\therefore \triangle\, DBF = \triangle\, ABE = \frac{1}{2} \triangle\, ABC$.

Ex. 37. If from any point in the diagonal of a parallelogram lines be drawn to the angles, the parallelogram will be divided into two pairs of equal triangles.

Given E in BD : let AC, BD cut at F : then $\triangle\ ABF = \triangle\ CBF$, and $\triangle\ AEF = \triangle\ CEF$; $\therefore \triangle\ AEB = \triangle\ CEB$, and $\triangle\ AED = \triangle\ CED$.

Ex. 38. Through E, the bisection of the diagonal BD of a quadrilateral $ABCD$, draw FEG parallel to AC ; and show that AG will bisect the figure.

$\triangle\ ABE = \triangle\ ADE$, and $\triangle\ CBE = \triangle\ CDE$, $\therefore$ figure $ABCE$ = figure $ADCE$: but $\triangle\ AEG = \triangle\ CEG$, $\therefore$ (if AG cut CE at H) $\triangle\ AEH = \triangle\ CGH$: from one figure take $\triangle\ AEH$, and add to it $\triangle\ CGH$, and *vice versâ* for the other figure ; $\therefore \triangle$ on one side AG = quadrilateral on the other.

Ex. 39. If of the four triangles, into which the diagonals divide a quadrilateral, two opposite ones are equal, the quadrilateral has two opposite sides parallel.

Let $\triangle\ AED = \triangle\ BEC$; $\therefore \triangle\ ABD = \triangle\ ABC$; $\therefore AB$ is $\parallel$ to CD.

Ex. 40. The two triangles, formed by drawing lines from any point within a parallelogram to the extremities of two opposite sides, are together half the parallelogram.

Through given point E, draw $FEG \parallel$ to AB ; then $\triangle\ AEB = \frac{1}{2} \square\ AG$, and $\triangle\ CED = \frac{1}{2} \square\ DG$; $\therefore \triangle\ AEB + CED = \frac{1}{2} \square\ ABCD$.

Ex. 41. If from the ends of one of the oblique sides of a trapezoid two lines be drawn to the bisection of the opposite side, the triangle thus formed with the first side is half the trapezoid.

Given E in BC: draw $FEG \parallel$ to AD: then $\triangle\ BEF = \triangle\ CEG$, and $\square\ AG$ = trapezoid = $2\ \triangle\ ADE$.

Ex. 42. If from the extremities of the base of an isosceles triangle lines be drawn perpendicular to the sides, the line which joins the vertex with their point of intersection will bisect the base at right angles.

BD, CD (*i.*) $\perp$ to AB, AC; (*ii.*) $\perp$ to AC, AB cutting AC, AB at E, F, in which case $\angle\ ABE = \angle\ ACF$; $\therefore$, for both cases, $\angle\ ABD \pm ABC = \angle\ ACD \pm ACB$, or $\angle\ DBC = \angle\ DCB$, and $DB = DC$; $\therefore \angle\ BAD = \angle\ CAD$, and AD bisects BC at rt. $\angle$s.

II. — Circles.

Ex. 1. The centre of a circle being given, find two opposite points in the circumference by means of a pair of compasses only.

Given O: with centre A (any point in circumference), radius AO, describe $\odot$, cutting the former at B, C; with centre B, radius BA or BO, describe $\odot$, cutting the first at D; then $\triangle$s COA, AOB, BOD are equilateral. $\therefore \angle$s COA, AOB, BOD = 2 rt. $\angle$s, and COD is a straight line, and = diameter of $\odot$.

Ex. 2. Through a given point draw a line, so that the part of it intercepted between two given parallel lines may be equal to a given line.

With centre A (any point in one of the lines) and radius = given line, describe $\odot$, cutting the other line at B, C: through given points draw lines $\parallel$ to AB, AC.

Ex. 3. If, with the vertex of an isosceles triangle as centre, a circle be described cutting the base or base produced, the parts of it intercepted between the circle and the extremities of the base will be equal.

Let the ⊙, centre A, cut BC at D, E; draw $AF \perp$ to BC: then $DF = FE$, and $BF = FC$; $\therefore DB = CE$.

Ex. 4. If two circles cut each other, any two parallel lines drawn through the points of section to cut the circles are equal.

Let the Ⓢ (centres A, B) intersect at C, D, and let ECF, GDH, be $\parallel$ lines; draw KAM, LBN, $\perp$ to EF, GH: then $EK = KC$, $CL = LF$; $\therefore EF = 2\,KL = 2\,MN = GH$.

Ex. 5. If two circles cut each other, draw through one of the points of section a line which shall be terminated at the circumferences, and be bisected at that point.

Let the Ⓢ (centres A, B) intersect at C, D: bisect AB at E, and draw $FCG \perp$ to CE, AH, $BK \perp$ to FG, EL, BM, $\perp$ to AH, EC: then $EL = BM$, or $CH = CK$; $\therefore CF = CG$.

Ex. 6. Draw a line cutting two concentric circles, so that the part of it intercepted by the circumference of the greater, may be double the part intercepted by that of the less.

Produce any radius OA, making $AB = OA$; on AB describe ⊙, cutting, when possible, outer ⊙ at C; draw $CADE$ the line required: for ($OF \perp$ to CE) by ◬ OAF, BAC, $AF = AC$ and $CE = 2\,AD$.

Ex. 7. If two circles cut each other, the greatest line that can be drawn through the point of intersection is that which is parallel to the line joining their centres.

Given $ECF \parallel$ to AB, GCH any other line through C: draw

AK, $BL \perp$ to EF, AM, $BN \perp$ to GH, $BP \perp$ to AM; then $GH = 2\,BP < 2\,AB < EF$.

Ex. 8. Describe three equal circles touching one another, and also another which shall touch all three.

On any line AB describe equilateral $\triangle ACB$; bisect BC, AC, AB at D, E, F; with centres A, B, C, radius AF, BD, CE; describe the Ⓢ. Again, bisect the $\angle$s of the $\triangle$ by AG, BG, CG, cutting the Ⓢ at H, K, L: then $AG = BG$; $\therefore GH = GK = GL$, and the $\odot$ with centre G radius GH, will touch the 3 Ⓢ at H, K, L, since AG, BG, CG pass through their centres.

N.B.—This is true either of the enveloping or the included $\odot$.

Ex. 9. How many equal circles can be described around another circle of the same magnitude, touching it and one another?

A the centre of central $\odot$, B, C of two others: then $\angle BAC = \frac{2}{3}$ rt. $\angle$, and there may be six such angles around the point A, and $\therefore$ six such Ⓢ.

Ex. 10. Describe a circle which shall pass through a given point, and touch a given circle at a given point, the two points not being in a tangent to the given circle.

Given A, and B in $\odot$, centre O: produce OB to C, and make $\angle BAC = \angle ABC$; C is the centre.

Ex. 11. Describe a circle which shall touch a given circle at a given point, and also touch a given straight line.

Draw tangent at A cutting given line BC at C; bisect $\angle ACB$ by CD, cutting OA at D; draw $DB \perp$ to BC; then $DB = DA$, and D is the centre.

Ex. 12. If from any point without a circle two lines be drawn, making equal angles with the line through the centre from that point, they will cut off equal segments from the circle.

From centre O draw OB, $OC \perp$ to given lines ADE, ACF; then $OB = OC$, and $\therefore DE = CF$.

Ex. 13. In the diameter of a circle produced determine a point from which a tangent drawn to the circle shall be equal to the diameter.

Given diameter AOB: draw $AC \perp$ to and $= AB$, and $DE \perp$ to ODC; then DE is a tangent, and $= AC = AB$.

Ex. 14. Describe a circle that shall pass through a given point, have a given radius, and touch a given line.

Given A, BC; draw $BD \perp$ to $BC =$ given radius, and let $DE \parallel$ to BC cut, if possible, the $\odot$, centre A and given radius at E; E is centre of $\odot$.

Ex. 15. Describe a circle whose centre shall be in the perpendicular of a given right-angled triangle, and which shall pass through the right angle and touch the hypotenuse.

Bisect $\angle B$, opposite to $\perp$, by BD, and draw $DE \perp$ to hypotenuse BC; then $DA = DE$, and D is centre of $\odot$.

Ex. 16. A is any point in the diameter (or diameter produced) of a circle whose centre is O; OB a radius perpendicular to the diameter: if AB cut the circle at P, and the tangent at P cut AO at C, show that $AC = CP$.

Since $\angle OPC =$ rt. $\angle$, we have $\angle s\ OPB + CPA =$ rt. $\angle$ $= \angle s\ OBP + CAP$; $\therefore \angle CPA = \angle CAP$, and $CP = CA$.

Ex. 17. A common tangent is drawn to two circles which touch externally: if a circle be described on that part of it which lies between the points of contact, as diameter, it will pass through the point of contact of the two circles, and be touched by the line joining their centres.

From point of contact C draw $CF \perp$ to AB cutting common tangent DE at F; then $FD = FC = FE$, and $\odot$ on DE will pass through C, and be touched by AB, since CF is $\perp$ to AB.

Ex. 18. Describe a circle, with given radius and its centre in a given line, which shall touch another given line.

Given AB, AC: from any point C in AC draw $CD \perp$ to AC and $=$ given radius, and let $DE \parallel$ to AC cut AB at E: E is the centre of $\odot$.

Ex. 19. Describe a circle that shall touch a given line at a given point, and also touch a given circle.

O the centre of given $\odot$: from given point A draw $AB \perp$ to given line, on side towards O, and in it or in BA produced, take $AC =$ radius of given $\odot$; make $\angle COB = \angle OCB$; then B is centre of $\odot$.

Ex. 20. Draw a line that shall touch a given circle, and make with a given line a given angle.

Given line AB and centre O: make $\angle BAC =$ given $\angle$, and through O draw $BC \perp$ to AC, and cutting $\odot$ at D; then $DE \parallel$ to AC is line required.

Ex. 21. Describe two circles of given radii, that shall touch each other, and the same given line on the same side of it.

From any point A in the line, and $\perp$ to it on the same side,

draw AB, AC = given radii; produce BA making $AD = AC$, and let ⊙, centre B, radius BD, cut $CE \parallel$ to given line at E: B, E are centres of Ⓢ required.

Ex. 22. If two circles touch each other, and parallel diameters be drawn, then lines which join the extremities of these diameters will pass through the point of contact.

Take Ⓢ touching externally: A, B the centres, C the point of contact, DE, FG, the $\parallel$ diameters: draw ACB, and join DC, CF; then $\angle EAC = \angle CBG$, $\therefore \angle ADC = \angle CFB$, and $\therefore DCF$ is a straight line; for, if not, produce DC to cut FG at F'; then $\angle ADC = \angle CF'B = \angle CFB$.

Ex. 23. The line drawn from the vertex of an equilateral triangle to meet the circumscribing circle at any point, is equal to the *sum* or *difference* of the two lines drawn from the extremities of the base to that point, according as it *does* or *does not* cut the base.

(*i.*) Let chord AD cut BC; in AD take $DE = DB$; then since $\angle BDE = \angle BCA$, and $DB = DE$, $\triangle BDE$ is equilateral; $\therefore \angle DBE = \angle ABC$, and $\angle DBC = \angle ABE$; $\therefore AE = CD$, and $AD = BD + CD$. (*ii.*) Let BD be the line, not cutting base AC: then, as before, $AB = BD + CD$, $\therefore BD = AD - CD$.

Ex. 24. The circles described on the three sides of a triangle, so as to pass through the points of intersection of the perpendiculars upon them from the opposite angles, are equal to each other.

Let the ⊥s Aa, Bb, Cc intersect at D; draw the diameters DE, DF, DG; then $\angle DCE =$ rt. $\angle DCF$, and $\therefore ECF$ is a straight line, and so are EBG, FAG; now $\angle CAa = \angle CBb$, $\therefore \angle CFD = \angle CED$, and $DF = DE = DG$.

Ex. 25. Two circles intersect at A, B, the centre of one being in the circumference of the other: draw any chord ACD cutting them both, and show that $CB = CD$.

Through O, the given centre, draw diameter AOE: then $\angle AOB = \angle ACB$, $\therefore \angle BOE = \angle BCD$, and $\triangle$s OBE, CBD are equiangular: but $\triangle\ OBE$ is isosceles, $\therefore CB = CD$.

Ex. 26. If from any two points in the circumference of a circle there be drawn two lines to a point in any tangent to the circle, they will make the greatest angle when drawn to the point of contact.

Given A, B, and CD tangent at C: let AD cut the $\odot$ at E; then $\angle ACB = \angle AEB > \angle ADB$.

Ex. 27. Given three points in a circle: show how we may find any number of other points, without knowing the position of the centre.

Given A, B, C: draw lines making $=\angle$s with AC, BC; they will meet at D, a point in the $\odot$.

Ex. 28. If, through the angles of a quadrilateral, lines bisecting them be drawn, the points at which each line intersects the adjacent ones will all lie in the circumference of a circle.

Given AGE, BGF, CHF, DHE, the bisecting lines: then $\angle$s $GEH + \frac{1}{2} A + \frac{1}{2} D = 2$ rt. $\angle$s $= \angle$s $GFH + \frac{1}{2} B + \frac{1}{2} C$: $\therefore$ $\angle$s $GEH + GFH + \frac{1}{2}(A + B + C + D) = 4$ rt. $\angle$s; but $\angle$s $A + B + C + D = 4$ rt. $\angle$s: $\therefore EHFG$ may be inscribed in a $\odot$.

Ex. 29. If two equal circles cut each other, and from either point of intersection a circle be described cutting them, the point where this circle cuts them, and the other

point of intersection of the equal circles are in the same straight line.

Let the ⊙s cut at A, B; with centre B describe ⊙ cutting them at C, D: let CA cut ⊙ ADB at E (suppose); then arc BE = arc BC, and $\therefore BE = BC$; $\therefore E$ is in ⊙ CD, and being also in ⊙ ADB, must be their point of intersection D.

Ex. 30. Given the radius of a circle that touches two given lines not parallel: determine its centre.

Bisect the $\angle BAC$ between the two lines by AO; draw $AD \perp$ to AB and = given radius: then a line through D $\perp$ to AD will cut AO at O, the centre of ⊙ required.

Ex. 31. Find a point in the diameter produced of a given circle, such that, if tangents be drawn from it to the circle, the concave part of the circumference may be double of the convex.

Produce radius OA making $AB = OA$; B is the point required: for on OB describe equilateral $\triangle\, OCB$, and let OC cut the ⊙ at D; then since $OD = CD$, BD is $\perp$ to OD, and $\therefore$ touches the ⊙; and $\angle AOD = \frac{2}{3}$ rt. $\angle$, $\therefore \angle A'OD = \frac{4}{3}$ rt. $\angle = 2\,\angle AOD$, and arc $A'D = 2$ arc AD.

Ex. 32. The line which is drawn through the bisection of any arc of a circle, parallel to its chord, is a tangent to the circle at that point: and the radius which bisects the chord of an arc bisects also the arc.

Given $DCE \parallel$ to chord AB, through C the bisection of its arc: let OC cut AB at F; then $\angle AOC = \angle BOC$; $\therefore$ by $\triangle$s AOF, BOF, $\angle$s at F are rt. $\angle$s; $\therefore$ also $\angle$s at C are rt. $\angle$s, and DE is a tangent at C. Again, let OC bisect AB at F; then by $\triangle$s ACF, BCF, $AC = BC$, and $\therefore$ arc AC = arc CB.

Ex. 33. If, from each extremity of two adjacent arcs of a circle, lines be drawn through two given points in the opposite circumference, and produced till they meet, the angles formed by these lines will be equal.

Given arc AB = arc BC, and D, E; let AD, BE meet at F, BD, CE at F'; draw chords EG, EH, $\parallel$ to AD, BD; then arc AG = arc DE = arc BH, $\therefore$ arc BG = arc CH, and $\angle BEG$ or $F = \angle CEH$ or F'.

Ex. 34. ACB, ADB are arcs of equal circles, on the same line AB, and on the same side of it: draw any chord ACD cutting them both, and show that BC and BD are equal.

If the arcs were on *different* sides of the line, they would make up the whole circle; hence $\angle s\ ACB + ADB$ $= 2$ rt. $\angle s = \angle s\ ACB + BCD$; $\therefore \angle BCD = \angle BDC$, and $BC = BD$.

Ex. 35. If circles be described on the two sides of a right-angled triangle as diameters, they will be touched by a circle whose centre is the bisection of the hypotenuse, and diameter equal to the sum of the sides.

Bisect hypotenuse BC at D, draw DEG, DFH bisecting AB, AC, at rt. $\angle s$ at E, F; then $AEDF$ is a $\square$, and DG $= DE + EG = AF + EA = FH + DF = DH$; $\therefore \odot$ with centre D, radius DG or DH, will touch the $\odot s$ at G, H, having diameter $= 2\,DG = 2\,AE + 2\,AF = AB + AC$.

Ex. 36. The circles described on the sides of any triangle as diameters will intersect in the sides, or sides produced, of the triangle.

Let $\odot s$ on AB, AC cut BC at D, E; then $\angle ADB =$ rt. $\angle$ $= \angle AEC$, which is impossible, unless D and E coincide.

Ex. 37. The vertical angle of any oblique-angled triangle, inscribed in a circle, is greater or less than a right angle, by the angle contained by the base and the diameter drawn from the extremity of the base: and no parallelogram can be inscribed in a circle except a rectangle.

$\angle BAC = \angle BAD \pm \angle CAD =$ rt. $\angle \pm \angle CBD$. Again the diameters of a ▱ bisect each other, and ∴ both pass through the centre; hence the ∠s of the figure must be each a rt. ∠.

Ex. 38. From one extremity of a line, which cannot be produced, draw a line perpendicular to it.

Given A: with any centre C, radius CA, describe ⊙, cutting line again at B: produce BC to D in this ⊙; AD is ⊥ to AB.

Ex. 39. If two circles touch each other, and any two lines be drawn passing through the point of contact, the chords of the intercepted arcs will be parallel.

Let any common chord through A cut the Ⓢ at B, C, and let the common diameter cut them at D, E; then $\angle BAD = \angle CAE$, and rt. $\angle ABD =$ rt. $\angle ACE$, ∴ $\angle ADB = \angle AEC$, and the segments cut off by the line are similar; ∴ if any other line through A cut the Ⓢ at B', C', since $\angle ABB' = \angle ACC'$, ∴ BB' is ∥ to CC'.

Ex. 40. Given one angle, the side opposite, and the sum of the other two sides: construct the triangle.

On given side AB describe a segment containing $\frac{1}{2}$ given ∠, and ∴ > $\frac{1}{2}$ ⊙, in which place $AC =$ given sum; make $\angle CBD = \angle BCA$; then $AB + DB = AC =$ given sum, and exterior $\angle ADB = 2 \angle ACB =$ given angle.

Ex. 41. Through three given points draw three lines so as to make an equilateral triangle.

Given A, B, C: on AC, BC describe segments containing each an $\angle = \frac{2}{3}$ rt. $\angle$; draw any line DCE cutting these circles; and draw DAF, CBF; the $\triangle\ DEF$ is equilateral.

Ex. 42. If two circles cut each other, draw through either point of section a line cutting both the circles and equal to a given line; and hence through three given points draw lines, so as to make a triangle equal in all respects to a given triangle.

Given A, B the centres, C a point of section: on AB describe $\odot$, in which place (if possible) $BD = \frac{1}{2}$ given line, and through C draw $ECF \parallel$ to BD cutting the ⓢ; draw ADH, $BG \perp$ to EF; then $EF = 2\ BD =$ given line. Again, if A, B, C be the 3 given points, on AC, BC describe segments containing $\angle$s = two $\angle$s of given $\triangle$, and draw DCE cutting them and = side adjacent to them: draw DAF, EBF; the $\triangle\ DEF$ is that required.

Ex. 43. Describe a circle which shall pass through a given point, and touch a given line at a given point.

Given A, and B in CD; draw $BE \perp$ to CD, and make $\angle BAF = \angle ABE$; F is the centre of the $\odot$.

Ex. 44. Given the perpendicular from the vertex of a triangle to the base, the difference of the segments of the base, and the sum of the two sides: to construct the triangle.

In AB, = sum of sides, take AC = difference of segments; draw $CD \perp$ to AC and = given altitude, and produce it so that $DE = CD$. From centre A, radius AB, describe $\odot\ BFG$. Find centre O of a $\odot$ passing through C and E, and touching $\odot\ BFG$ at F; join AO, CO; then $\triangle\ AOC$ has the sum of AO, $CO = AF$ or AB, as required, but its base is AC; $\therefore$ with centre O, and radius = the side not the

greater, describe ⊙ cutting AC at H; then OHC is the required Δ.

Ex. 45. Describe an isosceles triangle, having given the base angle and the perpendicular from it upon the opposite side.

On given $\perp$, AB, describe a segment containing given $\angle$; draw chord $BC \perp$ to AB; bisect AC at D, and draw DE $\perp$ to AC, to meet CB produced: EAC is the Δ.

Ex. 46. Given the vertical angle, the difference of the sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex: construct the triangle.

Take AB = difference of base segments; make $\angle ABC$ = $\frac{1}{2}$ given $\angle$, and draw AC = difference of sides; produce AC to D, and make $\angle CBD = \angle BCD$; with centre D, radius $DB = DC$, describe ⊙ cutting AB, AD, at E, F; ADE is the Δ, since $\angle ADE = 2\angle AFE = 2\angle ABC$.

Ex. 47. Given the vertical angle, the line drawn to the base bisecting that angle, and the difference between the base and the sum of the sides: construct the triangle.

Given $\angle BAC$, and AD; in AB, AC, take $AE = AF$ = $\frac{1}{2}$ given difference; draw EG, $FG \perp$ to AB, AC, and with centre G, radius $GE = GF$ describe ⊙, and through D draw BDC touching it at H; then $BE = BH$, $CF = CH$; ∴ difference of sides and base $= AE + AF =$ given difference.

Ex. 48. Given the angles of a triangle and the radius of the inscribed circle: construct the triangle.

OF = given radius; make $\angle FOB$ = rt. $\angle - \frac{1}{2}\angle B$; and

$\angle FOC =$ rt. $\angle - \frac{1}{2}\angle C$; draw $BFC \perp$ to OF, and make $\triangle OBA, OCA = \triangle OBF, OCF$; ABC is the Δ.

Ex. 49. Given the vertical angle of a triangle and the radii of the inscribed and circumscribed circles: construct the triangle.

Take $\angle BAC =$ given $\angle$, bisect it by $AO =$ radius of circumscribed $\odot$; describe $\odot$ with centre O, radius OA, cutting AB, AC at B, C; bisect BC at rt. $\triangle$ by $DE =$ radius of inscribed $\odot$, and with centre E, radius ED, describe $\odot$; draw BF, CF tangents to it, which must meet in circumference of former $\odot$ if the problem be possible; then BFC is Δ required.

III. — Ratios and Similar Figures.

Ex. 1. Through a given point between two given lines draw a line, such that the parts intercepted by it and the two given lines may be equal.

Given A, BC, BD: in BA produced take $AE = AB$; draw $ED \parallel$ to BC; then DAC is the line required.

Ex. 2. ABC is an equilateral triangle, E any point in AC; in BC produced take CD, CF equal to CA, CE, respectively, and let AF, DE, intersect at H: show that $HC : EC :: AC : AC + EC$.

CH bisects $\angle ACD$, $\therefore HC$ is $\parallel$ to AB, and $HC : CF :: AB : BF$, or $AC : EC :: HC : AC + EC$.

Ex. 3. If a square be inscribed in a right-angled triangle, one side coinciding with the hypotenuse, the base is divided in continued proportion.

Given ▭ $DFGE$, DE in BC: then by similar △s $BD : DF :: GE : EC$, or $DB : DE :: DE : EC$.

Ex. 4. If ABC be an inscribed triangle, and BD be drawn parallel to the tangent at A to meet AC, or AC produced, show that AB is a mean proportional between AC, AD.

$\angle EAB =$ alt.-int. $\angle ABD =$ also $\angle ACB$: $\therefore$ by similar △s, $AC : AB :: AB : AD$.

Ex. 5. AB is divided at C, D, so that $AB : AC :: AC : AD$; if AE be any other line taken equal to AC, show that the angle BED is bisected by EC.

For since $AC = AE$, $AB : AE :: AE : AD$; $\therefore$ △s AEB, ADE are similar, and $\angle ABE = \angle AED$; but $\angle ACE = \angle AEC$, and $\angle ACE =$ △s ABD, BEC; $\therefore \angle BEC = \angle CED$.

Ex. 6. The part of a tangent to a circle, intercepted by tangents at the extremities of any diameter, is divided at the point of contact so that the radius is a mean proportional between the two segments.

Given AC, BD intercepting CED; then since $\angle COD =$ rt. $\angle$, we have $CE : EO :: EO : ED$.

Ex. 7. If, in similar triangles, from any two equal angles, lines be drawn to the opposite sides, making equal angles with homologous sides, these will have the same ratio as the sides on which they fall, and will also divide them proportionally.

△s ABG, DEH are similar, as also △s CBG, FEH; $\therefore$ $BG : EH :: AB : DE :: AC : DF$; and $AG : DH :: BG : EH :: GC : HF$.

Ex. 8. In any triangle, right-angled at A, if CD be

drawn bisecting the angle C, show that $AB : AC :: BC - AC : AD$.

Draw $DE \perp$ to BC: then $CA = CE$, $DA = DE$; and by similar △s, $AB : AC :: EB : ED :: BC - AC : AD$.

Ex. 9. If two circles touch externally, the part of their common tangent between the points of contact is a mean proportional between the diameters.

EF the common tangent, $CD \perp$ to ACB; then $ED = CD = FD$, and $\angle ADB =$ rt. $\angle$; $\therefore AC : CD :: CD : BC$; $\therefore$ &c.

Ex. 10. Given two circles which intersect: draw through either point of intersection a line cutting the circles, so that the chords intercepted may be in a given ratio.

Divide AB at D in given ratio; and C being point of section, draw $ECF \perp$ to CD, and AG, $BH \perp$ to EF: then $EC : CF :: GC : CH :: AD : DB$.

Ex. 11. Inscribe in a given triangle a parallelogram similar to a given parallelogram.

Given ▱ $PQRS$: draw $AD \parallel$ to BC; make $\angle DAE = \angle P$; take $AD : AE :: PQ : PS$; let BD cut AC at q, draw qp, $qr \parallel$ to BC, AE, and complete ▱ $pqrs$: then $pq : AD :: Bp : BA :: ps : AE$; $\therefore pq : ps :: AD : AE :: PQ : PS$, and the ▱s are similar.

Ex. 12. If through the vertex and extremities of the base of a triangle any two circles be described so as to intersect in the base or base produced, their diameters will be proportional to their respective sides.

Draw the diameters AE, AF: then $\angle AEB = \angle ADC = \angle AFC$; $\therefore$ by similar △s, $AE : AF :: AB : AC$.

Ex. 13. Construct a triangle, having given one side, the angle opposite to it, and the ratio of the other two sides.

On given base BC describe segment containing given $\angle$, and draw $DE \perp$ to BC to cut remaining segment at E; divide BC at F in given ratio, and join EF cutting $\odot$ at A; ABC is the $\triangle$.

Ex. 14. If an isosceles triangle be inscribed in a circle, and from the vertical angle a line be drawn to meet the circumference and base, the rectangle of the segments of this line is equal to the square on either of the sides of the triangle.

Let ADE cut $\odot$ at D and BC at E; then $\angle ADC = \angle ABC = \angle ACE$; $\therefore$ by similar $\triangle$s, $AD : AC :: AC : AE$, or $AD.AE = AC^2$.

Ex. 15. Within a given circle place six equal circles touching one another and the given circle; and show that the interior circle which touches them all is equal to each of them.

Take AB the side of a regularly inscribed hexagon; produce OB, making $BC = \frac{1}{2} AB$: draw Ba, $ab \parallel$ to CA, AB: then by similar $\triangle$s, $ab : Bb :: AB : BC :: 2 : 1$; $\therefore$ ⊙s described with centres a, b, radii aA, bA, will touch each other and the given $\odot$; also $OB = AB = 2\,BC$, $\therefore Oa = 2\,Aa$, and $\odot$, centre O, radius $= Aa$, will touch them all.

Ex. 16. Find a point without a given circle, such that the sum of the two lines, drawn from it touching the circle, shall be equal to the line drawn from it through the centre to meet the circle.

Take diameter AOB; draw $AC \perp$ to $AB = \frac{1}{2} AO$, and

draw tangent DCE cutting AB at E; then by similar $\triangle$ $EA = \frac{1}{2} ED$, and $EA.ED = ED^2$, $\therefore EB = 2\,ED$.

Ex. 17. ABC is an isosceles triangle; draw AD perpendicular to the base, and DEF, cutting AB, AC, at E, F; then $AD : DE :: AB + AF : AB - AF$.

Draw $FG \perp$ to BC; then $ED : FG :: BD : BG$ and $FG : AD :: CG : CD$; $\therefore ED : AD :: CG : BG :: BD - DG : BD + DG :: BE - EF :: BE + EF :: BA - AF : BA + AF$.

Ex. 18. From any point P, tangents PA, PB are drawn to a circle, and AC is drawn perpendicular to the diameter BD: show that AC is bisected by PD at E.

By similar $\triangle$, $CE : CD :: PB : BD$, and $CD : CA :: OB : PB$; $\therefore CE : CA :: OB : BD :: 1 : 2$.

Ex. 19. AD is drawn bisecting the vertical angle of a triangle, and cutting the base BC at D; in BC produced take a point E, equally distant from A and D, and show that $BE : DE :: DE : CE$.

$\angle ACE = \angle CAD$, $ADC = \angle BAD$, $DAE = \angle BAE$; $\therefore$ by similar $\triangle$ $BE.EC = AE^2 = DE^2$, or $BE : DE :: DE : CE$.

Ex. 20. If through the bisection of the base of a triangle any line be drawn, cutting one side of the triangle, the other produced, and a line drawn parallel to the base from the vertex, this line shall be cut harmonically.

Let the line bisect BC at D, and cut AB, AC, and the line through $A \parallel$ to BC, at E, F, G; then $GE : GA :: DE : DB$, and $GA : GF :: DC : DF$; $\therefore GE : GF :: DE : DF$.

Ex. 21. If four diverging lines cut a straight line harmonically, they will cut any other intercepted line harmonically.

Let $BCDE$ be any other line; draw $Bcde \parallel$ to given line, and $\therefore$ also cut harmonically; and let $FDH \parallel$ to Be cut AC, AE, at F, H; then by similar $\triangle$s, $BE : Be :: DE : DH$, and $de : dc :: DH : DF$, and $Bc : BC :: DF : DC$, and (hyp.) $Be : Bc :: de : dc$; $\therefore BE : BC :: DE : DC$.

Ex. 22. If from the angle A of any parallelogram any line be drawn cutting the diagonal at E, and the sides BC, CD, at F, show that AE is a mean proportional between EF, EG.

By similar $\triangle$s, $EG : AE :: ED : EB :: AE : EF$.

Ex. 23. CAB, CEB, are two triangles which have a common angle B, and the sides CA, CE, equal: if, in BE produced, there be taken ED, a third proportional to BA, AC, then will the triangles BDC, BAC be similar.

$BA : AC :: AC$ (or EC) $: ED$; $\therefore \triangle$s BAC, CED are similar, and thence also $\triangle$s BAC, BDC may be shown to be similar.

Ex. 24. Construct an isosceles triangle equal to a given scalene triangle, and with the same vertical angle.

Produce BA, making $AD = AC$; on BD describe $\odot$, and draw $AE \perp$ to BD; then $AE^2 = BA.AD$; hence, if in AB, AC, we take AF, AG, each $= AE$, the $\triangle$s AFG, ABC will be equal.

IV. — AREAS.

Ex. 1. If two sides of a triangle be produced, the lines which bisect the two exterior angles and the third interior angle meet all at one point.

Produce CA, CB to F and G; bisect $\angle s$ FAB, ABG by AD, BD; draw $DE \perp$ to AB, and DF, $DG \perp$ to CA, CB. Prove $DE = DG$, and similarly $DE = DF$, $\therefore DF = DG$; then, since $DF^2 + FC^2 = CD^2 = DG^2 + GC^2$, $\therefore FC = GC$; hence $\angle FCD = \angle GCD$.

Ex. 2. Find a point in the diagonal produced of a square, from which if a line be drawn parallel to any side of the square, and meeting another side produced, it will form, with the produced diagonal and produced side, a triangle equal to the given square.

In AB produced take $AE =$ diagonal AC; let EF, $\parallel$ to BC, meet AC at F; draw $EG \perp$ to AF; then $\triangle AEF = 2\triangle AEG = 2\triangle ABC = ABCD$; $\therefore$ F is point required.

Ex. 3. The area of a trapezoid is half that of a parallelogram, whose base is the sum of the two parallel sides, and altitude the perpendicular distance between them.

$AD \parallel$ to BC; in BC produced take $CE = AD$; complete $\square\, ABEF$, and draw DG, $CH \parallel$ to AB; then $\square\, AG = \square\, HE$, and $\triangle DGC = \triangle DHC$; $\therefore$ figure $ABCD =$ figure $DCEF = \frac{1}{2} AE = \frac{1}{2} \square$ whose base $BE = AD + BC$, and altitude $=$ that of trapezoid.

Ex. 4. On the sides AB, AC of a triangle describe parallelograms $ABDE$, $ACFG$, and produce DE, FG to meet at H; then the area of these parallelograms together is

equal to the area of the parallelogram on BC, whose side is equal and parallel to AH.

Let BK, CL, ∥ to AH, meet DH, FH at K, L; join KL cutting AH at M; then AK, AL are ▱s, ∴ $BK = AH = CL$, and ∴ $BCLK$ is the ▱ on BC; now ▱ AD = ▱ AK = ▱ BM, and ▱ AF = ▱ CM.

Ex. 5. Upon a given base describe an isosceles triangle equal to a given triangle.

Given △ ABC; bisect BC at D, draw DE ⊥ to BC to meet a line through A ∥ to BC; EBC is the △ required.

Ex. 6. Show that the perimeter of an isosceles triangle is less than that of any other equal triangle upon the same base.

Let ABC be an isosceles △, DBC any other equal △, AD ∥ to BC; draw BE ⊥ to AD, and produce making $EF = BE$; join AF, DF; then CAF is a straight line, and $FD + DC > FC$, or $BD + DC > BA + AC$.

Ex. 7. From a given point in one of the equal sides of an isosceles triangle, draw a line meeting the other side produced, which shall make with these sides a triangle equal to the given triangle.

Given D in AB; draw BE to AC, ∥ to DC; then △ DCE = △ DCB; ∴ △ ADE = △ ABC.

Ex. 8. If one angle of a triangle be a right angle, and another be two-thirds of a right angle, show the equilateral triangle on the hypotenuse is equal in area to the sum of those on the sides.

Given ∠ A = rt. ∠, ∠ B = ⅔ rt. ∠, AFB, AEC, BDC the △s; then ∠ ABD = rt. ∠, and DCE is a straight line; draw

$CG, CH, AK \perp$ to BD, AE, BF; then $\triangle BDC = 2 \triangle CBG$ $= 2 \triangle ABC = 2 \triangle ACH + 2 \triangle AKB$ (since $\triangle ABC$ $= \frac{1}{2} \square BCHK) = \triangle AEC + AFB$.

Ex. 9. Convert a trapezoid into a triangle of equal area with one angle common; and hence show how to transform any rectilineal figure into a triangle, whose vertex shall be in a given angle of the figure and base in one of the sides.

$AB \parallel$ to CD; draw DE to BC, $\parallel$ to AC; then $\triangle ACD$ $= \triangle ACE$, and $\triangle ABE =$ figure $ABCD$.

Ex. 10. Given a triangle ABC and a point D in AB: construct another triangle ADE equal to the former, and having the common angle A.

Given D in AB; draw BE to AC, $\parallel$ to DC; then $\triangle DBC$ $= \triangle DEC$, and $\triangle ABC = \triangle ADE$.

Ex. 11. Change a triangle into another equal one of given altitude.

Draw $BD \perp$ to BC, $=$ given altitude; let DE, $\parallel$ to BC, cut AB at E; make $\triangle BEF = \triangle ABC$, and then *any* $\triangle$, on base BF, and with vertex in DE, will $= \triangle ABC$.

Ex. 12. In any given line, AB is taken half of AC: if through B, C parallel lines be drawn, cutting any other line through A at D, E, then AD is half of AE, and BD of CE, and the triangle ABD a fourth of the triangle ACE; and, conversely, if BD be such that AD is half of AE, then BD is parallel to CE.

Draw $DF \parallel$ to AC; then $DF = BC = AB$, and also $\parallel$ to AB, $\therefore BF$ is $\parallel$ to AE, and $AD = BF = DE$, $CF = BD$ $= FE$, and $\triangle ABD = \frac{1}{4} \triangle ACE$; conversely, since $AB = BC$,

$\triangle ABD = \triangle CBD$; but if $AD = DE$, $\triangle ABD = \triangle EBD$, which $\therefore = \triangle CBD$; hence BD is $\parallel$ to CE.

Ex. 13. If two exterior angles of a triangle be bisected, the line drawn from the point of intersection of the bisecting lines to the opposite angle of the triangle will bisect it.

Let lines bisecting ext. $\angle$s at B, C meet at D, and draw DE, DF, DG, $\perp$ to BC, AC, AB; then in $\triangle$s DBG, DBE, $DG = DE = DF$; also $AD^2 = AG^2 + DG^2 = AF^2 + DF^2$, $\therefore AG = AF$; $\therefore$ in $\triangle$s AGD, AFD, $\angle GAD = \angle FAD$.

Ex. 14. If a line be drawn from one of the acute angles of a right-angled triangle to the bisection of the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square upon half the line bisected.

Given rt. $\angle BAC$, and BD the line; then $BC^2 = BA^2 + AC^2 = BA^2 + 4\,AD^2 = BD^2 + 3\,AD^2$.

Ex. 15. If from the middle point of one of the sides of a right-angled triangle a perpendicular be drawn to the hypotenuse, the difference of the squares on the segments so formed is equal to the square on the other side.

Given $AD = DC$; draw $DE \perp$ to BC; then $BD^2 = BE^2 + ED^2$, $CD^2 = CE^2 + ED^2$; $\therefore BE^2 \pm CE^2 = BD^2 \pm CD^2 = BD^2 \pm AD^2 = AB^2$.

Ex. 16. Let AOB be a quadrant of a circle whose centre is O; from any point C in its arc draw CD perpendicular to OA or OB, meeting at E the radius which bisects the angle AOB; then show that the squares upon CD, DE, are together equal to the square upon OA.

$\angle DOE = \frac{1}{2}$ rt. $\angle = \angle DEO$, $\therefore DO = DE$, and $OD^2 + DC^2 = OC^2$, or $CD^2 + ED^2 = OA^2$.

Ex. 17. If from any point in the diameter of a semicircle two lines be drawn to the circumference, one to the bisection of the arc, and the other perpendicular to the diameter, then the squares upon these two lines are together double of the square upon the radius.

Given C in diameter, AOB, CD, CE the lines; then $CD^2 + CE^2 = (OC^2 + OD^2) + (OE^2 - OC^2) = 2\ OD^2$.

Ex. 18. If A be the vertex of an isosceles triangle ABC, and CD be drawn perpendicular to AB, prove that the squares upon the three sides are together equal to the square on BD, and twice the souare on AD. and thrice the square on CD.

$AB^2 + AC^2 + BC^2 = 2\ AC^2 + BC^2 = 2\,(AD^2 + CD^2) + (BD^2 + CD^2) = BD^2 + 2\ AD^2 + 3\ CD^2$.

Ex. 19. If from any point perpendiculars be dropped on all the sides of any rectilineal figure, the sum of the squares upon the alternate segments of the sides will be equal.

Take a $\triangle ABC$, and from P draw PD, &c. $\perp$ to AB, &c.; then $AD^2 + DP^2 = AP^2 = AF^2 + FP^2$, $BE^2 + EP^2 = BP^2 = BD^2 + DP^2$, $CF^2 + FP^2 = CP^2 = CE^2 + EP'^2$; $\therefore AD^2 + BE^2 + CF^2 = AF^2 + BD^2 + CE^2$.

Ex. 20. Divide, when possible, a given line into two parts, so that the sum of their souares mav be equal to a given square.

Given AB; make $\angle ABC = \frac{1}{2}$ rt. $\angle$, with centre A, radius = side of given square, describe $\odot$ cutting, when possible, BC at C, and draw $CD \perp$ to AB; then $AD^2 + DB^2 = AD^2 + DC^2 = AC^2$.

Ex. 21. From D, the middle point of AC, one of the

sides of an equilateral triangle ABC, draw DE perpendicular on BC; and show that the square upon BD is three-fourths of the square upon BC, and the line BE three-fourths of BC.

BD is $\perp$ to AC; $\therefore BD^2 = BC^2 - CD^2 = BC^2 - \frac{1}{4} BC^2 = \frac{3}{4} BC^2$; and since BE is $\perp$ to DEF, a side of the equilateral $\triangle$ on BD, we have $BE^2 = \frac{3}{4} BD^2 = \frac{9}{16} BC^2$; $\therefore BE = \frac{3}{4} BC$.

Ex. 22. Produce a given line so that the rectangle of the whole line produced and the original line shall be equal to a given square.

Given AB; draw $BD \perp$ to AB, and with centre A, radius = side of given square, describe $\odot$ cutting, if possible, BD at D, and draw $DE \perp$ to AD; then $EA.AB = AD^2$.

Ex. 23. If on the radius of a circle a semicircle be described, and a perpendicular to the common diameter be drawn, the square on the chord of the greater circle, between the extremity of the diameter and the point of section of the perpendicular, will be double of the square on the corresponding chord of the lesser circle.

AOB the common diameter; from C in OA draw CDE $\perp$ to AB; then $AE^2 = AB.AC = 2\,AO.AC = 2\,AD^2$.

Ex. 24. Divide a straight line into two parts so that the sum of their squares may be the least possible.

Bisect AB at C; then if D be another point in AB, $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$, and $\therefore AD^2 + DB^2$ is least, when CD is least, or D coincides with C, the bisection of AB.

Ex. 25. If a line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are

together equal to twice the rectangle contained by these parts, together with four times the square on the line between the points of section.

Given C the bisection, D any other point, of AB; then $AB^2 = AD^2 + DB^2 + 2\,AD.DB$, and also $= 4\,AC^2 = 2\,(AD^2 + DB^2 - 2\,CD^2)$; $\therefore AD^2 + DB^2 = 2\,AD.DB + 4\,CD^2$.

Ex. 26. If from one of the equal angles of an isosceles triangle a perpendicular be dropped on the opposite side, the rectangle of that side and the segment of it between the perpendicular and base is equal to half the square upon the base.

Draw $BD \perp$ to AC; then $AC^2 = AB^2 + BC^2 - 2\,AC.CD$: $\therefore AC.CD = \frac{1}{2}\,BC^2$.

Ex. 27. If the sides of a triangle be as 2, 4, 5, show whether it will be acute or obtuse angled.

Since 5^2 (or 25) $> 4^2 + 2^2$ (or 20), the $\angle$ opposite the side represented by 5 will be obtuse.

Ex. 28. If one angle of a triangle be four-thirds of a right angle, the square on the side subtending that angle is equal to the sum of the squares on the sides containing it, together with the rectangle contained by these sides.

Given $\angle BAC = \frac{4}{3}$ rt. $\angle$; draw $BD \perp$ to AC; $\therefore \angle BAD = \frac{2}{3}$ rt. $\angle$, and $AD = \frac{1}{2}\,AB$; $\therefore BC^2 = BA^2 + AC^2 + 2\,CA.AD = BA^2 + AC^2 + BA.AC$.

Ex. 29. If from the right angle of a right-angled triangle lines be drawn to the opposite angles of the square described on the hypotenuse, the difference of the squares on these lines is equal to the difference of the squares on the two sides of the triangle.

$BDEC$ the square on BC; produce DB, EC to meet at F, G, a line through $A \parallel$ to BC; then $AD^2 = AB^2 + BD^2 + 2\ BD.DF$, $AE^2 = AC^2 + CE^2 + 2\ CE.EG$; $\therefore AD^2 \pm AE^2 = AB^2 \pm AC^2$.

Ex. 30. Produce one side of a scalene triangle, so that the rectangle contained by it and the part produced may be equal to the difference of the squares on the other sides.

Given $\triangle\ ABC$; draw $CD \perp$ to AB, and produce AB to F, making $BF = AD - DB$; then $AC^2 - CB^2 = AD^2 - DB^2 = (AD + DB)(AD - DB) = AB.BF$.

Ex. 31. Any rectangle is half the rectangle contained by the diagonals of the squares upon its two sides.

Given AB, BC in one line; on AC describe square $ADEC$, and draw BF to $CD \parallel$ to AD; then $CD^2 = FD^2 + FC^2 + 2\ FC.FD$; or $2\ AC^2 = 2\ AB^2 + 2\ BC^2 + 2\ FC.FD$; but $2\ AC^2 = 2\ AB^2 + 2\ BC^2 + 4\ AB.BC$; $FC.FD = 2\ AB.BC$.

Ex. 32. If from any point within a rectangle lines be drawn to the angular points, the sums of the squares upon those drawn to the opposite angles will be equal.

Given E; let AC, BD, cut at F; then $AE^2 + CE^2 = 2\ AF^2 + 2\ EF^2 = 2\ BF^2 + 2\ EF^2 = BE^2 + DE^2$.

Ex. 33. The squares on the diagonals of any quadrilateral are together double of the squares on the two lines joining the bisections of the opposite sides.

E, &c., the bisections of AB, &c.; then $EFGH$ is $\square$, and $AC = 2\ EF$, $BD = 2\ EH$, $\therefore AC^2 + BD^2 = 4\ EF^2 + 4\ EH^2 = 2\ EF^2 + 2\ GH^2 + 2\ EH^2 + 2\ FG^2 = 2\ EG^2 + 2\ FH^2$.

Ex. 34. If DE be drawn parallel to the base BC of an

isosceles triangle ABC, then the square on BE is equal to the rectangle of BC, CD, together with the square on CE.

Draw DF, $EG \perp$ to BC; then $BE^2 = EC^2 + BC^2 - 2\,BC.CG = EC^2 + BC\,(BC - 2\,CG) = EC^2 + BC.DE$.

Ex. 35. The squares on the diagonals of a trapezoid are together equal to the squares on its two parallel sides, with twice the rectangle contained by its parallel sides.

Draw AE, $BF \perp$ to DC; then (E, F, both *within* DC) $AC^2 = AD^2 + CD^2 - 2\,CD.DE$, $BD^2 = BC^2 + CD^2 - CD.CF$; $\therefore AC^2 + BD^2 = AD^2 + BC^2 + 2\,CD\,(CD - DE - CF) = AD^2 + BC^2 + 2\,CD.AB$.

Ex. 36. If two points be taken in the diameter of a circle equally distant from the centre, the sum of the squares on two lines drawn from these points to any point in the circumference will be constant.

Given points C, D in diameter AOB; then, E being in circumference, $CE^2 + DE^2 = 2\,OC^2 + 2\,OE^2$, which is constant.

Ex. 37. Two parallel chords in a circle are respectively six and eight inches in length, and are one inch apart: how many inches in length is the diameter?

OBD is $\perp$ from centre O on semichords AB, CD; $OD = x$; then $OC^2 = x^2 + 9 = OA^2 = (x-1)^2 + 16$; $\therefore x = 4$ and diameter $= 10$ in.

Ex. 38. Given the area and hypotenuse of a right-angled triangle: construct it.

Bisect hypotenuse AB at C; on BC describe a rectangle = given area; and on AB describe $\odot$ cutting side of rectangle at D; then $\triangle\,ADB = 2\,\triangle\,CDB =$ rectangle on BC = given area.

Ex. 39. Given the area, one angle, and a line drawn from one of the others, bisecting the opposite side: construct the triangle.

On given line AB describe a segment containing given $\angle$, and also a rectangle = given area, whose side (when possible) cuts the segment at C; in CA produced make $AD = AC$; BCD is the Δ required.

Ex. 40. Describe a circle which shall touch a given line, and pass through two given points on the same side of the line.

Join A, B; and if the given line be $\parallel$ to AB, bisect AB at rt. $\angle$s by CD, meeting given line at D, and make $\angle DAE = \angle ADC$; E is the centre required: but if not, let AB produced meet the given line at C; and on given line take CD the side of a square $= AC.CB$, the required $\odot$ will pass through the points A, B, D.

Ex. 41. Describe a circle that shall touch a given circle, and pass through two given points that are both within or both without the given circle.

Through A, B, the two given points, describe $\odot$ cutting given $\odot$ at D and E; join AB, DE, and if these are $\parallel$, a $\perp$ from the middle point of AB will pass through the required point in given circumference: but if not, produce to meet at F, and draw FC a tangent to given $\odot$. The $\odot$ through A, B, C is the one required, because $AF.FB = DF.FE = FC^2$.

Ex. 42. If from the centre of a circle a line be drawn to any point in the chord of an arc, the square on that line, together with the rectangle of the segments of the chord, will be equal to the square on the radius.

Given centre O, and C in chord AB; draw $DCE \perp$ to OC; then $AC.CB = DC^2 = OD^2 - OC^2$; $\therefore OC^2 + AC.CB = OD^2$.

Ex. 43. If there be any three circles in a plane, and, through the centres of every two of them a circle be described touching the third, the lines joining the centre of each of the circles with the point at which the circles passing through it intersect, will meet at the same point.

Given A, B, C the centres, and a, b, c the points of intersection; let Aa, Bb, cut at D; and let CD (suppose) cut arc Aa at E, arc Bb at F; then $CD.DE = AD.Da - BD.Db = CD.DF$; $\therefore E$ and F coincide at c.

Ex. 44. The circle described through any two of the angular points of a triangle and the intersection of the perpendiculars from the angles on the opposite sides, will be equal to the circumscribing circle of the triangle.

Describe $\odot$ through B, C, G; then ($BD.CE$ is $\perp$ on BA, AB) $\angle s\ EAD + EGD = \angle s\ BAC + BGC = 2$ rt. $\angle s$; $\therefore$ remaining segment of $\odot\ BGC$ contains an $\angle = \angle\ BAC$, and being on same base BC must $=$ segment BAC; the whole $\odot\ BGC =$ whole $\odot\ BAC$.

V.—Regular Figures.

Ex. 1. The square on the side of an equilateral triangle inscribed in a circle, is triple the square on the side of the regular hexagon inscribed in the same circle.

Bisect $\angle\ ABC$ by diameter BOD; then $AD =$ side of hexagon $= DO$; and $AB^2 = BD^2 - AD^2 = 3\ AD^2$.

Ex. 2. Inscribe a square in a given right-angled isosceles triangle.

Trisect hypotenuse BC at D, E, and complete rectangle $DFGE$; then, since $\triangle\, DBF$ is equiangular to $\triangle\, ABC$, $DF = DB = DE$, and $DFGE$ is a square.

Ex. 3. Through two given points describe a circle touching a given circle; and show that, of all lines that can be drawn from the two points to meet the convex circumference, those drawn to the points of contact thus obtained will contain the greatest possible angle.

Through A, B describe any $\odot$ cutting given $\odot$ at C, D; let AB, CD meet at E, and draw E, F touching given $\odot$; then $EF^2 = EC.ED = EA.EB$, and the $\odot$ described through A, B, F will touch the line EF, and $\therefore$ the given $\odot$. Again, take any other point C in the given $\odot$; and let AC cut the tangent at F at G; then $\angle AFB > \angle AGB < \angle ACB$.

Ex. 4. The area of an inscribed regular hexagon is three-fourths that of the one circumscribed about the same circle.

AB side of inscribed hexagon; draw tangents AC, BC, and AD, BD bisecting rt. $\angle$s OAB, OBA; let ODC cut AB at E; then $\angle DAO = \angle DOA$, and $\angle DAC = \angle DCA$; $\therefore$ $DO = DA = DC$, and $CE = \frac{1}{2}CD = \frac{1}{4}\, OC$; $\therefore \triangle\, AOE = \frac{3}{4} \triangle\, AOC$, &c.

Ex. 5. Upon a given line as diagonal describe a rhombus, so that two of its angles shall be double that of the other two. Hence show how a right angle may be trisected.

Bisect AB at C, and on AC describe equilateral $\triangle\, ADC$, about which describe $\odot$; draw $CE \perp$ to AB; then $\angle AEC = \angle ADC = \frac{2}{3}$ rt. $\angle = 2 \angle EAC$, and AE, EB are sides of

the rhombus; also drawing $AF \perp$ to AB, the rt. $\angle BAF$ is trisected by AD, AE.

Ex. 6. The centres of the inscribed and circumscribed circles of an equilateral triangle coincide, and the diameter of one is double that of the other.

The line bisecting $\angle A$ bisects also BC, and if $OA = OB = OC$, we have also $OD = OE = OF$; also $AO = 2\ OD$.

Ex. 7. The lines joining the alternate angles, or the intersections of the alternate sides, of a regular pentagon, will form another regular pentagon.

Join $AabC$, $BbcD$, &c.; then $\angle BAC = \angle ABE$, $\therefore Aa = Ba$, and $\angle Aac = 2 \angle ABE = 2 \angle AEB = \angle Aea$; $\therefore Aa = Ae = Ba = Ee$, and the $\triangle$s Aae, Bab, &c., are isosceles and equilateral, $\therefore ae = ab =$ &c., and $\angle eab = \angle abc =$ &c. Again, let EA, CB meet at A', AB, DC at B', &c.; then, since $AB = AE$, &c., and $\angle$s at base are $=$, the $\triangle$s $A'AB$, $B'BC$, are equal and equilateral; $\therefore$ $\triangle$s $A'BB'$, $A'AE'$, &c., are equal and equilateral; $\therefore A'B = A'E'$, &c., and $\angle E'A'B' = \angle A'B'C'$, &c.

Ex. 8. Inscribe in a given circle a rectangle equal to a given rectilineal figure.

On diameter AB describe rectangle $ABCD =$ given figure; let CD cut, when possible, the $\odot$ at E; then rectangle $AEBF = 2 \triangle AEB =$ rectangle $ABCD =$ given figure.

Ex. 9. Inscribe the least possible square in a given square.

If $AA' = BB' =$ &c., the figure $A'B'C'D'$ is a square, and $A'C'^2 = 2\ A'B'^2$; now $A'C'^2$, and $\therefore A'A'^2$ is least when the sides are bisected, or $A'C' = AB$.

Ex. 10. Describe a circle which shall pass through one angle and touch two sides of a given square.

Given $\angle A$ and BC, CD; bisect $\angle$s BAC, DAC by AE, AF, and draw EO, $FO \perp$ to BC, CD; then external $\angle COE$ ($= \frac{1}{2}$ rt. $\angle$) $= \angle$s OAE, OEA, and $\angle OAE = \frac{1}{4}$ rt. $\angle$, $\therefore$ $\angle OEA = \frac{1}{4}$ rt. $\angle$, and $OE = OA = OF$.

Ex. 11. If $ABCDE$ be a regular pentagon, show that the angles ABE, BCA, CDB, DEC, EAD, are together equal to two right angles.

These $\angle$s = each the $\angle$ at *circumference* on one side of figure; $\therefore$ together they = $\frac{1}{2}$ $\angle$s at *centre* on the 5 sides = 2 rt. $\angle$s.

Ex. 12. Given a regular pentagon: describe a triangle of the same area and altitude.

Draw BF, $EG \parallel$ to AC, AD; then $\triangle AFC = \triangle ABC$, $\triangle AGD = \triangle AED$, $\therefore \triangle AFG =$ pentagon $ABCDE$.

Ex. 13. If two diagonals of a regular pentagon be drawn cutting one another, the larger segments will be each equal to a side of a pentagon.

Let AC, BD cut at F; then external $\angle AFB = \angle$s at circumference on arcs AB and $CD = \angle BDE$; $\therefore AC$ is $\parallel$ to DE, and, similarly, AE is $\parallel$ to BD; and $\therefore AF = DE = AE = DF$.

Ex. 14. Divide a right angle into five equal parts.

Given rt. $\angle BAC$; construct an isosceles $\triangle BAD$, with $\angle$s A, D, double of $\angle B$; and divide $\angle BAD$, by double bisection, into 4 = $\angle$s.

Ex. 15. The opposite sides of a regular hexagon are

parallel: and if any two sides of an inscribed hexagon are parallel to two other sides, the remaining two will also be parallel.

By $\triangle$s $AFE, BCD, \angle FAE = \angle CBD, \therefore \angle BAE = \angle ABD$ = &c., and $ABDE$ is equiangular, and $\therefore$ a $\square$: hence AB is $\parallel$ to DE. Again, since AB is $\parallel$ to $DE, \angle ABE = \angle BED$; and since AF is $\parallel$ to CD, arc $AC =$ arc DF, and $\angle ABC = \angle DEF$; $\therefore \angle CBE = \angle BEF$, and BC is $\parallel$ to EF.

Ex. 16. Inscribe a regular hexagon in a given equilateral triangle, and compare its area with that of the triangle.

Trisect AB, BC, CA at D, E, F, G, H, K; the figure $DEFGHK$ is a regular hexagon, whose area $= \frac{2}{3}$ of that of the triangle.

Ex. 17. Let AB, CD, two alternate sides of a regular polygon, be produced to meet at E: show that the figure $AECO$ can be inscribed in a circle, O being the centre of the polygon.

Draw OF, $OG \perp$ to AB, CD; then $\angle COG = \angle AOF$, $\therefore \angle$s $AOC, AEC = \angle$s $FOG, FEG =$ 2 rt. $\angle$s.

Ex. 18. If R, r, be the radii of circles described about and in a regular polygon, and R', r', the corresponding radii for a regular polygon of same perimeter, and twice the number of sides, show that $r' = \frac{1}{2}(R + r)$, and $R'^2 = Rr'$.

Given side AB of first polygon; bisect it at rt. $\angle$s by COD, cutting circumscribed $\odot$ at D, and draw OE, $OF \perp$ to AD, BD; then since AD, BD are bisected, $EF = \frac{1}{2} AB$, and is a side of the second polygon; $\therefore$ if DO cut EF at G, $r' = DG = \frac{1}{2} DC = \frac{1}{2}(DO + OC) = \frac{1}{2}(R + r)$, and $R'^2 = DE^2 = DO.DG = Rr'$.

GEOMETRICAL EXERCISES

WITHOUT

SOLUTIONS.

GEOMETRICAL EXERCISES

WITHOUT SOLUTIONS.

I. — Rectilinear Figures.

Ex. 1. Prove that the sum and difference of two lines are together equal to twice the greater line.

Ex. 2. AB is bisected at O, and P is any point produced through A or B; prove that $\frac{1}{2}(AP + BP) = OP$.

Ex. 3. If ten lines meet in a point, and make equal angles with other, find each angle.

Ex. 4. Of two supplementary angles, the greater is four times the less. Find each angle.

Ex. 5. Prove that the bisectors of the four angles which one straight line makes with another form two straight lines perpendicular to each other.

Ex. 6. How many diagonals can be drawn in a polygon of 20 sides?

Ex. 7. Find the sum of the angles in a polygon of 8 sides; of 10 sides; of 12 sides; of 20 sides.

Ex. 8. If the polygons mentioned in the preceding exercise are equiangular, find the value of each angle.

Ex. 9. Two angles of a triangle are together twice as great as the third; find all three angles.

Ex. 10. In a polygon of 12 sides each successive angle is $1^\circ\ 20'$ more than the preceding; find all the angles.

Ex. 11. Find the angles of a quadrilateral in which three angles are equal, and each of these three times as great as the fourth.

Ex. 12. The exterior angle of an equiangular polygon is one-third of a right angle; find the number of sides in the polygon.

Ex. 13. In an isosceles triangle the angle at the vertex is one-fourth one of the base angles; find the three angles of the triangle.

Ex. 14. Enumerate all the cases in which *three* given parts out of the six parts of a triangle determine the triangle. Also the cases in which three given parts do not determine the triangle.

Ex. 15. Construct a triangle having given two sides and the angle opposite the less side. How many solutions are there to this problem? When is the problem impossible?

Ex. 16. If in a right triangle one of the acute angles is twice the other, the hypotenuse is twice the smaller leg.

Ex. 17. State and prove the *converse* to the preceding exercise.

Ex. 18. The line that joins the vertex to the middle point of the base of a triangle is less than half the sum of the three sides.

Ex. 19. If any two points of two parallel lines are joined by a straight line, and this line is then bisected, every straight line drawn through the point of bisection and between the parallels is also bisected at this point.

Ex. 20. Draw a straight line through a given point, so that it shall make a given angle with a given straight line.

Ex. 21. Draw a straight line through a given point, so that the parts intercepted between the points and the sides of a given angle shall be equal.

Ex. 22. In an equilateral triangle construct another, such that one of its vertices shall be a given point in one of the sides of the first triangle.

Ex. 23. Construct an isosceles triangle such that its vertex shall be at a given point, its base shall have a given length and shall lie on a given straight line.

Ex. 24. Two parallel lines and a point are given; construct an isosceles triangle having the point for vertex, and such that the extremities of its base shall lie in the parallel lines, while the base itself shall make a given angle with the parallel lines.

Ex. 25. Two lines are given in length and position; find a point which shall be the common vertex of isosceles triangles constructed upon the two lines as bases.

Ex. 26. Place three lines of given lengths together so that they shall coincide at one end, and the other ends shall lie in a straight line at equal distances from each other.

In a triangle ABC, let A, B, and C denote the angles, a, b, and c respectively the opposite sides, h the altitude CD upon AB taken as base, p and q the two parts BD and AD into which AB is divided by the point D. Also let m denote the line drawn from C to the middle point E of AB, w the length of the bisector of the angle C measured from C to the point F where it cuts AB; lastly, let $BF = u$, $AF = v$. If the triangle is a right triangle, let C be the right angle; if obtuse-angled, let A be the obtuse angle. When necessary let it be understood that a is greater than b.

Construct a triangle having given —

Ex. 27. a, b, h.
28. b, h, B.
29. b, h, p.
30. b, h, C.
31. p, q, h.
32. h, A, B.
33. h, c, B.
34. h, c, b.
35. $a + b, h, B$.
36. $a - b, h, A$.
37. h, m, c.
38. p, m, b.

Ex. 39. h, w, C.
40. h, w, a.
41. m, b, c.
42. m, c, B.
43. w, B, C.
44. w, b, A.
45. $p - q, a, B$.
46. $p - q, a, b$.
47. $p - q, a, A$.
48. $a, b, A - B$.
49. $a, A - B, p - q$.
50. $p - q, A, B$.

NOTE. — In Exercises 45–50, A is obtuse, q must be regarded as negative, and $p + q$ must be written in place of $p - q$.

Construct a right triangle having given —

Ex. 51. h, a.

52. h, p.

53. h, A.

54. $a + b, c$.

55. $a + b, A$.

56. $c + b, a$.

Ex. 57. $c + b, A$.

58. $a - b, c$.

59. $a - b, A$.

60. $c - b, a$.

61. $c - b, A$.

62. $a + b + c, A$.

Construct a triangle having given —

Ex. 63. $a + b, c, C$.

64. $a + b, c, A$.

65. $a + b, A, C$.

66. $a + b, c, A - B$.

67. $a - b, c, B$.

68. $a - b, c, C$.

69. $a - b, c, A$.

70. $a - b, A, B$.

71. $a - b, c, A - B$.

72. $c + b, h, B$.

73. $c - b, a, h$.

74. $a + b, A - B, C$.

75. $a + b + c, A, B$.

76. $a + b + c, A - B, C$.

77. $a + b + c, h, B$.

Ex. 78. $a + c - b, A, B$.

79. $a + c - b, h, A$.

80. $a + h, b, B$.

81. $a - h, A, B$.

82. $a, h, p - q$.

83. $a + h, p - q, B$.

84. $a + b, p - q, B$.

85. $a + b, p - q, C$.

86. $a + b, p - q, A - B$.

87. u, v, A.

88. $u, v, A - B$.

89. $u, v, a - b$.

90. $u, B, a - b$.

91. $u, A, a - b$.

NOTE. — In order to solve Exercises 27–91, draw a triangle, and assume it to be the triangle required; then find from the given data an auxiliary triangle in which three parts such that they determine the triangle shall be known. Construct this auxiliary triangle by the ordinary rules; then construct the triangle required.

The following auxiliary triangles will be found useful: —

1. On DB take $DG = DA$; $\triangle BCG$ has the parts $a, b, p-q$, $180^\circ - A, B, A-B$.

2. Produce BC making $CH = CA$; $\triangle BHA$ has the parts $a+b, c, 90^\circ + \frac{1}{2}(A-B), \frac{1}{2}CB$.

3. On CB take $CJ = CA$; $\triangle BJA$ has the parts $a-b$, c $\frac{1}{2}(A-B), 90^\circ + \frac{1}{2}C, B$.

4. Produce CA making $AK = a-b$; $\triangle BAK$ has the parts $a-b, c, \frac{1}{2}(A-B), 90^\circ - \frac{1}{2}C, 180^\circ - A$.

5. $\triangle BHG$ has the parts $a+b, p-q, 90^\circ + \frac{1}{2}C, \frac{1}{2}(A-B), B$.

6. $\triangle BJF$ has the parts $u, v, a-b, 180^\circ - A, B, A-B$.

Ex. 92. If from a point in the base of an isosceles triangle perpendicular lines are drawn to the two sides, their sum is equal the altitude of the triangle if one of these two sides is taken as base.

Ex. 93. If from a point inside an equilateral triangle perpendicular lines are drawn to the three sides, their sum is equal to the altitude of the triangle.

Ex. 94. The bisectors of the four angles which the diagonals of a parallelogram make with each other meet the sides of the parallelogram in points which are the corners of a rhombus.

Ex. 95. The middle points of the sides of a rectangle are the corners of a rhombus.

Ex. 96. The middle points of the sides of a rhombus are the corners of a rectangle.

Ex. 97. If the middle points of the sides of any quadrilateral, taken in succession, are joined by straight lines, the resulting figure is a parallelogram.

Ex. 98. Place a line of given length between the sides of a given angle so that it shall be parallel to a given line.

Ex. 99. Find a point in one side of a triangle from which the lines drawn to the other sides, and parallel to them respectively, shall be equal.

Ex. 100. A trapezoid whose non-parallel sides are equal is divided by the two diagonals into four triangles, of which the two having for bases the non-parallel sides are equal, and the other two are isosceles.

In a trapezoid $ABCD$, let a denote the longer of the two parallel sides BC, b the shorter, AD; also let $AB = c$, $DC = d$, $AC = e$, $BD = f$, the altitude of the trapezoid $= h$; let, also, A, B, C, D denote the angles of the trapezoid, m the acute angle of the diagonals, n the angle CBD. Construct a trapezoid having given —

Ex. 101.	$abcB$.	Ex. 116.	$nBad$.	Ex. 131.	$hacf$.
102.	$acdA$.	117.	$nBae$.	132.	$hacC$.
103.	$acBC$.	118.	$nBce$.	133.	$hacm$.
104.	$abcf$.	119.	$nCab$.	134.	$hcde$.
105.	$abeB$.	120.	$nCde$.	135.	$haef$.
106.	$abeC$.	121.	$nBCa$.	136.	$hefB$.
107.	$acfB$.	122.	$nBCf$.	137.	$hamn$.
108.	$cdeC$.	123.	$mnac$.	138.	$hcmn$.
109.	$aeBC$.	124.	$mnae$.	139.	$hBCn$.
110.	$aefB$.	125.	$mnBc$.	140.	$abcd$.
111.	$nacd$.	126.	$mnBa$.	141.	$abcC$.
112.	$nace$.	127.	$habc$.	142.	$adBC$.
113.	$nacf$.	128.	$habe$.	143.	$nBcd$.
114.	$naef$.	129.	$habn$.	144.	$a-b, c, d, n$.
115.	$nBac$.	130.	$hacd$.	145.	$a-b, h, d, f$.

Construct a quadrilateral having given —

Ex. 146. Three sides and the angle opposite to the fourth side.

147. Three angles and two adjacent sides.

148. Three sides and the two diagonals.

149. One side, the middle points of the other three sides, and a straight line to which the given side must be parallel.

Ex. 150. Construct a pentagon having given the middle points of its five sides.

Ex. 151. Cut off the corners of a square by lines which, with the sides of the square, shall form a regular octagon.

II. — Circles.

Ex. 152. The less the distance of a chord from the centre of a circle, the greater the chord.

Ex. 153. Of all the chords which can be drawn through a point within a circle, the shortest is that which is bisected by the point.

Ex. 154. The line joining the points of tangency of two parallel tangents passes through the centre of the circle.

Ex. 155. Parallel chords drawn from the extremities of a diameter are equal; and the line which joins the extremities of these chords passes through the centre of the circle.

Ex. 156. If, in the greater of two concentric circles, a chord is drawn which cuts the smaller circle, or touches it, the parts of the chord between the circles are equal.

Ex. 157. The difference between the sum of the two legs of a right triangle and the hypotenuse is equal to the diameter of the circle inscribed in the right triangle.

Ex. 158. The locus of the middle point of a line which moves with its ends touching two perpendicular lines is the circumference of a circle. What is the radius of this circle? Where is its centre?

Ex. 159. If a rectangle is inscribed in a circle, and through its corners tangents to the circle are drawn, these tangents form a rhombus.

Ex. 160. The perpendiculars let fall from the vertices of a triangle, bisect the angles of the triangle whose vertices are the points in which the perpendiculars meet the sides of the first triangle.

Ex. 161. If a circle is circumscribed about an equilateral triangle, and any point of its circumference joined to the vertices of the triangle by straight lines, the longest of these lines is equal to the sum of the two other lines.

Ex. 162. If a circle is circumscribed about any triangle, and from any point of the circumference perpendiculars are drawn to the three sides of the triangle, the points where they meet the sides are in one straight line.

Ex. 163. Construct a circle of given radius and passing through two given points. (Two solutions. In what case only one? In what case no solution?)

Ex. 164. Construct a circle of given radius and touching a given straight line at a given point.

Ex. 165. Construct a circle of given radius and tangent to a given straight line. (When is there *one*, when *two* solutions?)

Ex. 166. Construct a circle which shall pass through three given points. (What position must the points not have?)

Ex. 167. Construct a circle which shall pass through a given point and be tangent to a given line at a given point of the line.

Ex. 168. Construct a circle which shall pass through two given points and be tangent to a given line parallel to the line which joins the given points.

Ex. 169. Find the locus of the middle points of all chords in a circle that have a given length.

Ex. 170. Find the locus of points from which tangents drawn to a given circle shall have a given length.

Ex. 171. Through a given point in a circle draw the shortest chord.

Ex. 172. In a given circle draw a chord parallel to a given straight line, and having a given length.

Ex. 173. Find the locus of the centre of a circle which shall have a given radius and shall cut from a given straight line a chord of given length.

Ex. 174. Inscribe an equilateral triangle in a given circle.

Ex. 175. Inscribe in a given circle a triangle having angles equal to those of a given triangle.

Ex. 176. Through a given point draw a chord of given length in a given circle.

Ex. 177. Draw a tangent to a given circle parallel to a given straight line.

Ex. 178. Find a point from which tangents drawn to two given circles shall be equal in length.

Ex. 179. Find the locus of the middle points of all chords which can be drawn through a given point in the circumference of a circle.

Ex. 180. Three lines, A, B, C, are given: find a fourth, X, such that the lines AX, BX, CX shall make two equal angles with each other.

Ex. 181. Three points are given: find a fourth such that lines joining it to the given points shall make equal angles with one another.

Ex. 182. Construct a right triangle having given the middle point of the hypotenuse, the vertex of the right angle, and the length of one leg.

Ex. 183. Find the locus of the centre of a circle which shall be tangent to two given intersecting lines. (Two straight lines. What relative position do they have? Why?)

Ex. 184. Find the locus of the centre of a circle which shall be tangent to two given parallel lines.

Ex. 185. Construct a circle which shall touch the sides of a given angle, and whose centre shall lie in a given straight line.

Ex. 186. Construct a circle which shall touch two given intersecting straight lines, and shall touch one of these lines in a given point. (Two solutions. What position of the given point is excluded?)

Ex. 187. Construct a circle which shall have a given radius and shall touch two given intersecting lines.

Ex. 188. Construct a circle which shall be tangent to two given parallel lines, and shall pass through a given point between them. (Two solutions.)

Ex. 189. Construct a circle which shall be tangent to three given straight lines. (Four, two, or no solutions.)

Construct a right triangle with a given hypotenuse having also given —

Ex. 190. A line in which the vertex of the right angle must lie.

191. The altitude of the triangle with hypotenuse as base.

192. The point where this altitude meets the hypotenuse.

193. The distance of the vertex of the right angle from a given line.

Construct a triangle having given one side, the angle opposite, and also —

Ex. 194. The altitude with the given side as base.

195. The altitude with one of the other sides as base.

196. The length of the line which joins the middle point of the given side with the opposite vertex.

197. The sum of the other two sides.

Let a, b, c denote the sides of a triangle, A, B, C the angles opposite to these sides respectively, h the altitude of the triangle upon a as base, r the radius of the circumscribed circle. Construct a triangle having given —

Ex. 198. r, a, b.

199. r, a, B.

200. r, a, h.

201. r, A, B.

202. r, A, h.

203. $r, b, B - C$.

Ex. 204. $r, c, B - C$.

205. $r, a, B - C$.

206. $r, b + c, B$.

207. $r, b - c, B$.

208. $r, b - c, C$.

(In analyzing the above exercises make use of the fact that when a chord of a circle is given, the value of an angle inscribed in the corresponding segment is known; and conversely.)

Also construct a triangle having given —

Ex. 209. $r, a, b + c$.

210. $r, a, b - c$.

211. $r, A, b + c$.

Ex. 212. $r, A, b - c$.

213. $r, A, a + b + c$.

In a quadrilateral, $ABCD$, about which a circle can be circumscribed, let A, B, C, D denote the angles, a, b, c, d the sides opposite these angles respectively, e and f the diagnals AC and BD, r the radius of the circumscribed circle,

m the angle of the diagonals. Construct the quadrilateral having given —

Ex. 214.	$rabc$.	Ex. 219.	$raCD$.	Ex. 224.	$rabe$.
215.	$rabA$.	220.	$race$.	225.	$raem$.
216.	$racA$.	221.	$rabf$.	226.	$raAm$.
217.	$raAB$.	222.	$raef$.	227.	$refm$.
218.	$raAD$.	223.	$raeA$. (Compare Ex. 173.)		

Ex. 228. $r, a, c, B + C$. (Analysis: By means of r, a and c, two inscribed angles are determined, which, added to the angle of the segment corresponding to d, are equal to $B + C$.)

229. r, e, A, m. (Analysis: By means of A, f is determined. Compare Ex. 227.)

230. $bceA$. (Analysis: by means of A, C is determined; by means of b, c and C the circle is determined.)

231. $abef$. Ex. 232. $abeA$. Ex. 233. $abfA$.

234. $abBm$. 235. $aceB$. (Analysis: by means of e and B the circle is determined.)

Ex. 236. $eABm$. Ex. 237. $abem$.

Construct a quadrilateral in which a circle can be inscribed, using the preceding notation, except that r shall now denote the radius of the inscribed circle, having given —

Ex. 238.	$raeA$.	Ex. 242.	$raBC$.	Ex. 246.	$abcB$.
239.	$rabB$.	243.	$reAC$.	247.	$abcA$.
240.	$raAC$.	244.	$rabA$.	248.	$abeA$.
241.	$reAB$.	245.	$abce$.	249.	$abBC$.

Let the quadrilateral be a trapezoid, a the greater, b the less, of the parallel sides, and construct it having given —

Ex. 250. rbd. Ex. 251. $r, a + c, b$. Ex. 252. rbc.

Let the trapezoid have the non-parallel sides equal, and let h be its altitude; construct it having given—

Ex. 253. ac.	Ex. 255. ch.	Ex. 257. bA.
254. bc.	256. bh.	258. cC.

Ex. 259. The locus of the centres of all circles which touch a given circle at a given point is the line passing through the given point and the centre of the given circle. What two points of this line cannot be centres of the circles?

Ex. 260. The locus of the centres of all circles which touch a given circle, and have a given radius, consists of two circles concentric with the given circles. What are the radii of these circles? In what case does one of them disappear?

Ex. 261. The locus of the centres of all circles which touch two given concentric circles consists of two circles concentric with the given circles. What are the radii of these two circles? What is the simplest way to construct them?

Ex. 262. If two circles touch each other, every line drawn through the point of contact cuts from the circles arcs which subtend equal angles at the centre; and also the radii drawn from the ends of these lines are parallel lines. What difference does it make whether the circles have *external* or *internal* contact?

Ex. 263. If two equal circles cut one another, the arcs cut off are equal, and the common chord bisects the line of centres. Also the converse theorem.

Ex. 264. Unequal circles cut from one another unequal arcs. Also the converse theorem.

Ex. 265. Two circles cannot so cut each other as to be mutually bisected.

Ex. 266. Every line drawn through the point of intersection of the line of centres and the common chord of two circles which cut each other, and limited either by two interior or by two exterior arcs, is bisected by this point of intersection, and the radii drawn from its extremities are parallel lines. Also conversely.

Ex. 267. If from one of the points of intersection of two circles which cut each other a diameter of each circle is drawn, the other ends of these diameters lie in a straight line which passes through the other point of intersection of the circles.

Ex. 268. About a given point as centre describe a circle which shall touch a given circle. (When two solutions? When only one?)

Ex. 269. About two given points as centres describe circles which shall touch each other, and one of which shall touch a given straight line. (Analysis: first construct the circle which shall touch the line, and then make use of *Ex.* 268.)

Ex. 270. With a given radius describe a circle which shall touch a given circle, and pass through a given point. (Use the method of loci. Examine the cases where the given point is (*i.*) in, (*ii.*) inside, (*iii.*) outside the circumference of the given circle; also the cases where the radius

of the required circle (*i.*) is equal to, (*ii.*) is greater than, (*iii.*) is less than the radius of the given circle.)

Ex. 271. With a given radius r describe a circle which shall touch a given circle (radius $= R$), and also a given straight line. (Use method of loci. Special cases: r is equal to, is greater than, is less than R; distance of centre of given circle from the given line is equal to, is greater than, is less than $R + 2r$; in the last case, this distance is equal to, is greater than, is less than R.)

Ex. 272. Construct a circle which shall have a given radius a, and shall touch two given circles, radii R and r. (Use method of loci. Different cases: line of centres of the given circles $> R + r + 2a, = R + r + 2a, < R + r + 2a$; and in the last case the same line $> R + r, = R + r, < R + r$, etc.)

Ex. 273. Construct a circle which shall touch the two given radii and the arc of a given sector.

Ex. 274. Construct circles about the vertices of a given triangle such that each circle shall touch the other two circles.

(The points where the inscribed circle touches the sides of the given triangle are also the points of contact of the circles required.)

Ex. 275. Construct the locus of the centre of a circle which has a given radius, and cuts a given chord from a given circle.

Ex. 276. Construct a circle which has a given radius, passes through a given point, and cuts a diameter from a given circle.

Ex. 277. Construct a circle which has a given radius, touches a given straight line, and cuts a chord of given length from a given circle.

Ex. 278. Construct a circle which has a given radius, touches a given circle, and cuts a line of given length from a given straight line.

III. — Areas.

Ex. 279. Of all triangles of equal altitude which can be constructed upon a given line as base, the isosceles has the least perimeter.

Ex. 280. The rectangle whose sides are the hypotenuse and corresponding altitude of a right triangle is equal to the rectangle whose sides are the two legs of the right triangle.

Ex. 281. If the diagonals of a quadrilateral cut each other at right angles, the sum of the squares of one pair of opposite sides is equal to the sum of the squares of the other pair.

Ex. 282. In a rhombus the sum of the squares of the two diagonals is equal to the square of one-half the perimeter.

Ex. 283. In every parallelogram the sum of the squares of the two diagonals is equal to the sum of the squares of the four sides.

Ex. 284. If lines are drawn from the vertices of a triangle to the middle points of the opposite sides, four times

the sum of their squares is equal to three times the sum of the squares of the sides of the triangle.

Ex. 285. The locus of a point, the sum of the squares of whose distances from two fixed points is constant, is the circumference of a circle.

Ex. 286. Convert a parallelogram into another having a given angle.

Ex. 287. Convert a given triangle into another having (*i.*) a given side, (*ii.*) a given angle.

Ex. 288. Draw through the sides of a triangle a line which shall cut off a triangle equal to a given triangle.

Ex. 289. Convert a given triangle into an isosceles triangle.

Ex. 290. Convert a given triangle into a parallelogram having a given side.

Ex. 291. Convert a hexagon into a triangle.

Ex. 292. Convert a pentagon into a rectangle, one side of which shall lie in a given straight line.

Ex. 293. Convert a triangle into another having a given altitude.

Ex. 294. Convert a triangle into another having one side in common with the first triangle, and opposite to this side a given angle.

Ex. 295. Convert two given triangles into one equivalent triangle.

Ex. 296. Convert several given triangles into one equivalent triangle.

Ex. 297. Construct a parallelogram equal to a given triangle, and having the same perimeter.

Ex. 298. Halve a triangle by a straight line drawn from a given point in one of its sides.

Ex. 299. Halve a parallelogram by drawing a straight line which shall pass through a given point.

Ex. 300. Construct a square equal to one-half of a given square.

Ex. 301. Find a point within a triangle such that lines drawn from the point to the vertices of the triangle shall divide the triangle into three equal parts.

Ex. 302. From a given point within a triangle draw lines which shall divide the triangle into three equal parts.

Ex. 303. Halve a quadrilateral by a straight line drawn from one corner.

Ex. 304. Divide a parallelogram into four equal parts by drawing lines from a given point in one side.

Ex. 305. Find the perimeter and the area of a rectangle, having given —

(*i.*) base = 18^m; altitude = 24^m.
(*ii.*) base = 58.3^m; altitude = 20.4^m.
(*iii.*) base = 86.74^m; altitude = 54.9^m.
(*iv.*) base = 845.6^m; altitude = 1843.02^m.

Ex. 306. The floor of a rectangular room is 16^m long and 8^m wide. How many boards 4^m long and 0.5^m wide will be required to cover it?

Ex. 307. A rectangular field is 246.8^m long, and has 30751.28 square metres; how wide is it?

Ex. 308. The altitude of a rhombus $= 12.4^m$, the area $=$ 473.68 square meters; find one side.

Ex. 309. The perimeter of a rectangle $= 24.59^m$, and the base is double the altitude; find the area.

Ex. 310. The sum of two adjacent sides of a rectangle $=$ 8.2^m, their difference $= 1.4^m$; find the area.

Ex. 311. Find the area of a square whose side $= 26.12^m$.

Ex. 312. Find the area of a triangle if the base $= 56.8^m$, and the altitude $= 80.7^m$.

Ex. 313. Find the area of a right triangle whose legs are 248.2^m and 160.5^m.

Ex. 314. If the base of a triangle $= 8.4^m$, what altitude must it have in order to have the same area as a square whose side $= 5.6^m$?

Ex. 315. Find the area of a rhombus if its diagonals are 8.52^m and 6.38^m.

Ex. 316. The parallel sides of a trapezoid are 83.2^m and 110.4^m, the altitude is 50.4^m; find the area.

Ex. 317. The perimeter of a trapezoid $= 122^m$, the non-parallel sides are 36^m and 32^m, and the altitude $= 30.4^m$; find the area.

Ex. 318. The altitude and the parallel sides of a trapezoid are to each other as $2:3:5$, the area $= 1270.08$ square metres; find the altitude and the parallel sides.

Ex. 319. The perimeter of a rectangle is 13^{m} greater than one side, the area $= 20.88$ square metres; find the sides.

Ex. 320. The side of a square $= a$; find the side of a square whose area is n times as great.

Ex. 321. The sum of the areas of two squares $= 900$ square metres, the difference of their squares $= 252$ square metres; find the side of each.

Ex. 322. The sides of three squares are 8.4^{m}, 17.3^{m}, and 25.86^{m}; find the side of a square equal to their sum.

Ex. 323. The sides of two squares differ by 12^{m}, and their areas differ by 240 square metres; find the side and the area of each.

Ex. 324. The perimeter of a right triangle $= 60^{m}$, the hypotenuse $= 25^{m}$; find both legs and the area.

Ex. 325. In a right triangle given the perimeter P, and the hypotenuse h; find the area.

Ex. 326. The hypotenuse of a right triangle $= 26.5^{m}$, the difference of the legs $= 5.3^{m}$; find both legs and the area.

Ex. 327. Find the area of an isosceles triangle whose base $= a$, and side $= b$.

Ex. 328. The perimeter of an isosceles triangle $= 24^{m}$, and one of the equal sides $= 7.5^{m}$; find the area.

Ex. 329. Find the base of an isosceles triangle if the area $= 20.28$ square metres, and the side $= 6.5^m$.

Ex. 330. The hypotenuse of an isosceles right triangle $=$ 8.4^m; find the area.

Ex. 331. The area of an isosceles right triangle $=$ 270.0488 square metres; find its three sides.

Ex. 332. The hypotenuse of an isosceles right triangle is 15^m longer than one of the legs; find the area of the triangle.

Ex. 333. Find the area of an equilateral triangle one side of which $= 13.6^m$.

Ex. 334. If a denote one side of an equilateral triangle, find the area.

Ex. 335. If h denote the altitude of an equilateral triangle, find the area.

Ex. 336. One side of an equilateral triangle is 4^m greater than the altitude; find the side, the altitude, and the area.

Ex. 337. Find the area of a triangle whose sides are 585^m, 488^m, and 137^m.

Ex. 338. Two sides of a triangle are 3^m and 8^m, and the included angle is $60°$; find the area.

Ex. 339. Two sides of a triangle are 13^m and 15^m, and the angle opposite the first side is $60°$; find the area.

Ex. 340. Two sides of a triangle are 5^m and 19^m, and the angle opposite the second side is $60°$; find the area.

Ex. 341. One side of a triangle is 8^m, and the altitudes corresponding to the other sides are 6^m and 3^m; find the other two sides and the area.

Ex. 342. The three altitudes of a triangle are 10^m, 12^m, and 14^m; find the area.

Ex. 343. The base of a rectangle $= 306^m$, the altitude $= 229.5^m$; find the diagonals and the area.

Ex. 344. The sum of the two diagonals and one side of a square $= 100^m$; find the area.

Ex. 345. The perimeter of a square is 48^m greater than the sum of the diagonals; find the area.

Ex. 346. The area of a rectangle $= 2883$ square metres, the sum of the diagonals $= 77.5^m$; find the sides.

Ex. 347. Find the area of a rhombus if the altitude $= 48^m$, and one diagonal $= 60^m$.

Ex. 348. One side of a rhombus $= 3.44^m$, and one of the adjacent angles $= 60^C$; find the area.

Ex. 349. A side of a rhombus $= 20^m$, and a diagonal $= 30^m$; find the area.

Ex. 350. The area of a parallelogram $= 120$ square metres, and two adjacent sides are 12^m and 14^m; find the two diagonals.

Ex. 351. The two altitudes of a parallelogram are 5^m and 8^m, and one diagonal is 10^m; find the area.

Ex. 352. The parallel sides of a trapezoid are 184^m and 68^m, and the other sides are 84^m and 72^m; find the area.

Ex. 353. The diagonals of a trapezoid are 110^m and 100^m, and the distance between the parallel sides is 80^m; find the area.

IV.—Similar Figures.

Find by construction the value of x in the following expressions, in which a, b, c, etc., denote known lengths,—

Ex. 354. $x = \dfrac{3a + 2b}{5}.$

355. $x = \dfrac{ab}{c}.$

356. $x = \dfrac{a^2}{b}.$

357. $x = \sqrt{ab}.$

Ex. 358. $x = \dfrac{abcd}{fgh}.$

359. $x = \dfrac{a^2 + b^2}{c}.$

360. $x = \dfrac{ac + bc}{d + f}.$

361. $x = \dfrac{ab + cd}{f}.$

Ex. 362. $x = \sqrt{\dfrac{a^3 - b^3}{c}}.$

363. $x = \sqrt{a^2 + \sqrt{(b^4 + c^4)}}.$

364. $x = \sqrt{a^2 - b^2 + c^2 - df + gh}.$

Ex. 365. Find the geometric mean between the sum and the difference of two given lines.

Ex. 366. Construct two lines having given (*i.*) their sum and their geometric mean; (*ii.*) their difference and their geometric mean.

Ex. 367. Divide a line into two parts whose squares shall be to each other as two given lines.

Ex. 368. Find the locus of a point which moves so that the difference of the squares of its distances from two fixed points is always equal to a given square.

Ex. 369. Construct on the diagonal of a given rectangle an equal rectangle.

Ex. 370. Construct a right triangle having given —

(*i.*) The hypotenuse and the ratio $m : n$ of the legs.

(*ii.*) The altitude and the ratio $m : n$ of the legs.

(*iii.*) The altitude and the ratio $m : n$ of the segments of the hypotenuse.

Ex. 371. Construct an isosceles triangle having given the two altitudes.

Ex. 372. Construct a triangle having given —

(*i.*) The altitude, an angle at the base, and the ratio $m : n$ of the other two sides.

(*ii.*) A side, an adjacent angle, and the ratio $m : n$ of the other sides.

(*iii.*) The three altitudes.

Ex. 373. Construct a rhombus having given —

(*i.*) The area and the altitude.

(*ii.*) The area and one side.

(*iii.*) The area and a diagonal.

Ex. 374. Construct a trapezoid having given the parallel sides and the diagonals.

Ex. 375. Through a point inside a circle draw a chord of given length.

Ex. 376. Inscribe in a circle a triangle similar to a given triangle.

Ex. 377. Construct a circle which shall pass through a given point between the sides of an angle, and shall touch the sides of the angle.

Ex. 378. Construct a circle which shall pass through two given points and touch a given straight line.

Ex. 379. A circle and two external points are given: construct another circle which shall pass through the points and touch externally the given circle.

Ex. 380. Convert a square into a rhombus one side of which is given.

Ex. 381. Convert an isosceles triangle into an equilateral triangle.

Ex. 382. Convert a triangle into an equilateral triangle.

Ex. 383. Construct a square equal to three-fifths of a given square.

Ex. 384. Convert a square into an equilateral triangle.

Ex. 385. Convert a polygon into another polygon similar to a given polygon.

Ex. 386. Construct an equilateral triangle equal to five times a given equilateral triangle.

Ex. 387. Construct a triangle similar to a given triangle and twice as large.

Ex. 388. Construct a polygon similar to a given polygon and one-eighth as large.

Ex. 389. Divide a triangle, by lines from one vertex, into three parts which shall be to each other as $2:3:4$.

Ex. 390. Halve a quadrilateral by a line drawn from one corner.

Ex. 391. Divide a triangle, by lines parallel to one side, into (*i.*) two, (*ii.*) three, (*iii.*) four equal parts.

Ex. 392. Divide a triangle, by lines parallel to one side, into parts which shall be to each other as $2:3:5$.

Ex. 393. Halve a trapezoid by a line parallel to the parallel sides.

Ex. 394. In an equilateral triangle construct another half as large, and whose vertices fall in the sides of the given equilateral triangle.

Ex. 395. In a square construct another three-fourths as large, and the corners of which fall in the sides of the given square.

Ex. 396. In a semicircle construct a square, one side of which coincides in position with the diameter, and two corners of which fall in the circumference.

Ex. 397. The sides of a triangle are 17.4^m, 23.4^m, and 31.8^m; the shortest side of a similar triangle is 5.8^m; find the other two sides.

Ex. 398. The length of the shadow cast by a tree is found to be 37.8^m; find the height of the tree if a vertical rod 2.75^m long casts a shadow 1.4^m long.

Ex. 399. The altitude of a right triangle, the hypotenuse being taken as the base, divides the hypotenuse into two segments of 7.2^m and 16.2^m; find the altitude and the area of the triangle.

Ex. 400. The hypotenuse of a right triangle $= 36.5^m$, the sum of the legs $= 51.1^m$; find the legs and the altitude upon the hypotenuse as base.

Ex. 401. The perimeter of a triangle $= 4.37^m$; the sides of a similar triangle are 4.55^m, 6.3^m, and 4.445^m; find the sides of the first triangle.

Ex. 402. The area of a triangle $=$ 3259.6 square metres, one side $= 112.4^m$; find the area of a similar triangle if the side which corresponds to the given side of the first triangle $= 28.1^m$.

Ex. 403. The three sides of a triangle are 389.2^m, 486.5^m, and 291.9^m, and the area of a similar triangle is 2098.14 square metres; find the sides of the last triangle.

Ex. 404. The five sides of a pentagon are 12^m, 20^m, 11^m, 15^m, and 22^m; the perimeter of a similar pentagon is 16^m; find its sides.

Ex. 405. Two sides, AB and AC, of a triangle are 240^m and 270^m. If the triangle is divided into two equal parts by a line DF parallel to BC, find AD and AF.

Ex. 406. The three sides of a triangle, ABC, are $AB = 655^m$, $BC = 1075^m$, $AC = 860^m$. If a portion, ADF, containing 73960 square metres, is cut off by a line DF parallel to BC, find the distances AD, AF, and DF.

Ex. 407. A triangle, ABC, whose sides are $AC = 80^m$, $AB = 70^m$, $BC = 50^m$, is to be divided by lines perpendicular to AC into three equal parts; find the lengths of these lines of division, and the distances from C of the points where they cut AC.

Ex. 408. If the triangle of the preceding exercise be divided by lines perpendicular to AC into three parts, which, beginning with the part nearest C, are to each other as $4:5:7$, find where the lines of division cut AC.

Ex. 409. A triangular field, ABC, one side of which $AB = 100^m$, and the corresponding altitude $CD = 80^m$, consists of a trapezoid $ABHG$ and a triangle GHC, the altitude of the latter being 60^m. The first is worth \$3.00 per square metre, the second \$1.50. Divide the field, by a line parallel to AB, into two parts of equal value.

Ex. 410. A rectangular field, $ABCD$, whose sides are $AB = 200^m$, and $BC = 150^m$, is divided by the diagonal AC into two triangular parts differing in value. A square metre of the part ADC is worth \$3.00, and a square metre of the part ABC is worth \$2.00. Divide the field, by a line parallel to AB, into two parts of equal value.

V. — Regular Figures.

Ex. 411. The side of a circumscribed triangle is equal to twice the side of the inscribed triangle.

Ex. 412. The side of a regular circumscribed hexagon is two-thirds the side of the regular inscribed hexagon.

Ex. 413. The area of a circle circumscribed about an equilateral triangle is equal to four times the area of the circle inscribed in the same triangle.

Ex. 414. The area of a concentric ring is equal to that of a circle having for its diameter a chord of the greater circle, such that it touches the smaller circle.

Ex. 415. Given a circle; circumscribe and inscribe (*i.*) a regular octagon, (*ii.*) a regular decagon.

Ex. 416. A regular polygon with n sides being given, construct a regular polygon with $2n$ sides, and having the same perimeter.

Ex. 417. A regular polygon with n sides being given, construct a regular polygon with $2n$ sides, and having the same area.

Ex. 418. Make a circle whose circumference shall be equal to the sum of several given circumferences.

Ex. 419. Make a circle whose circumference shall be equal to the difference of two given circumferences.

Ex. 420. Make a circle whose area shall be equal to the sum of the areas of several given circles.

Ex. 421. Make a circle whose area shall be equal to the difference of the areas of two given circles.

Ex. 422. Make a circle whose circumference shall be to a given circumference as $m:n$.

Ex. 423. Make a circle whose area shall be to the area of a given circle as $m:n$.

Ex. 424. Halve a circle by describing a concentric circle.

Ex. 425. Divide a circle by describing concentric circles into three parts which shall be to each other as 2 : 3 : 5.

Ex. 426. The radius of a circle is 8^m; find the side of the inscribed equilateral triangle.

Ex. 427. With what radius must a circle be described, that the side of the inscribed equilateral triangle may be equal to 8.66^m?

Ex. 428. The radius of a circle is 6^m; find the side of the inscribed regular decagon.

Ex. 429. What radius must a circle have in order that the side of the inscribed regular decagon may be 9.27^m?

Ex. 430. The side of a regular dodecagon is 12.94^m; find the radius of the circumscribed circle.

Ex. 431. Find the side of a regular octagon in terms of R, the radius of the circumscribed circle.

Ex. 432. The side of a regular octagon is 1.4^m; find the radius of the circumscribed circle.

Ex. 433. Find the side of a regular pentagon in terms of R, the radius of the circumscribed circle.

Ex. 434. The side of a regular pentagon is 9.4^m; find the radius of the circumscribed circle.

Ex. 435. The side a of a regular polygon and the radius R of the circumscribed circle are given; find in terms of a

and R, the side x of the regular inscribed polygon which has half the number of sides of the given polygon.

Ex. 436. The radius of a circle is R; find the side S of the circumscribed regular polygon when this polygon is —

(*i.*) A triangle.	(*iv.*) A hexagon.
(*ii.*) A quadrilateral.	(*v.*) An octagon.
(*iii.*) A pentagon.	(*vi.*) A decagon.

Ex. 437. Find the side of a circumscribed regular polygon if the side of the inscribed regular polygon is 5^m.

Ex. 438. Find an expression for the radius R of the circle inscribed in the polygons of Exercise 436 if S denote a side of the polygon.

Ex. 439. The side S of a regular polygon is given; find the area A when the polygon is —

(*i.*) A triangle.	(*iv.*) A hexagon.
(*ii.*) A quadrilateral.	(*v.*) An octagon.
(*iii.*) A pentagon.	(*vi.*) A decagon.

Ex. 440. Find the area A of the polygons in Exercise 439 in terms of R, the radius of the circumscribed circle.

Ex. 441. Find the area A of the polygons in Exercise 439 in terms of r, the radius of the inscribed circle.

Ex. 442. Find the area of a regular decagon if one side $= 38^m$.

Ex. 443. The area of a regular octagon is 98.01 square metres; find the radius of the circumscribed circle.

Ex. 444. Find the area of a regular dodecagon if the radius of the inscribed circle $= 2.4^m$.

Ex. 445. With what radius must a circle be described in order that the area of the inscribed regular pentagon may be equal to that of a square whose side is 8.4^m?

Ex. 446. Compute the side of a regular hexagon if the area = 515.29 square metres.

Ex. 447. The radius of a circle circumscribed about a regular pentagon is 4.6^m; find the area of the pentagon.

Ex. 448. The circumference of the trunk of a tree = 4.7124^m; find the thickness of the tree and the area of a section.

Ex. 449. What is the diameter of a carriage wheel which makes 3125 revolutions in passing over a distance of $19{,}625^m$?

Ex. 450. The radius of a circle = 1.5^m; find the length of an arc of $78°$.

Ex. 451. The area of a circle = 78.54 square metres; find the length of the arc of $48° \ 12'$.

Ex. 452. Find the radius of a circle if an arc of $112°$ is 4^m longer than the radius.

Ex. 453. The diameter of a circle = 11.5^m, and the arc between two radii = 4.6^m; find the number of degrees in the arc.

Ex. 454. Find the number of degrees in an arc which is equal to the radius of the circle.

Ex. 455. Find the area of a sector if the angle at the centre = $68° \ 36'$, and the radius = 7.2^m.

Ex. 456. If the area of a circle $= 432$ square metres, find the area of a sector corresponding to the angle $84^\circ\ 12'$.

Ex. 457. Find the area of a segment which lies between the side of an inscribed regular pentagon and the corresponding arc, the perimeter of the pentagon being 15.708^m.

Ex. 458. The circumferences of two concentric circles are 21.98^m and 18.84^m; find the area of the ring included between them.

Ex. 459. Find an expression for the area A of a portion of a concentric ring, if R and r are the radii of the circles, and m the corresponding angle at the centre.

VI. — Solid Geometry.

Ex. 460. Find the exterior surface of a right prism with a square base, if the height $= 25.4^m$, and one side of the base $= 2.5^m$.

Ex. 461. A triangular prism is 18^m high, and the sides of the base are 2.4^m, 3^m, and 1.8^m; find the exterior surface.

Ex. 462. The sides of the base and the height of a rectangular parallelopiped are to each other as $2:3:15$; find their values if the whole exterior surface is 364.5 square metres.

Ex. 463. The diagonal of the square base of a right prism is 20^m less than the height of the prism. Find the values of both when the entire exterior surface is 278.5 square metres.

Ex. 464. Find the entire surface of a cube whose edge $= 5.8^m$.

Ex. 465. If the entire surface of a cube is 326.7864 square metres, find the edge.

Ex. 466. A ditch is to be dug 430^m long, 4^m deep, and 3^m wide; how many cubic metres of earth must be excavated?

Ex. 467. The height of a triangular prism $= 25^m$, one side of its regular base $= 1.8^m$; find its volume.

Ex. 468. Find the volume of a hexagonal column whose height is 32^m, and the perimeter of whose base is 5.4^m.

Ex. 469. If the volume of a prism with square base is 48.654 cubic metres, and its height is 8.5^m, find a side of the base.

Ex. 470. The surface of a rectangular parallelopiped = 664 square metres, the breadth is 2^m, and the length 6^m more than the height; find the volume.

Ex. 471. The three intersecting edges of a rectangular parallelopiped whose surface = 184.5 square metres are to each other as 2 : 3 : 7; find the volume.

Ex. 472. A tank with rectangular base contains 1316.25 cubic metres, and is 4.5^m deep; find the sides of the base if they differ by 2^m.

Ex. 473. A right triangular prism 5^m high weighs 1836^k; find one side of the regular base, the specific gravity of the material being 4.

Ex. 474. Required: the weight of 24 iron bars each 2^m long and having a square section whose side $= 8^{mm}$. (Specific gravity of iron, 7.8.)

Ex. 475. The perimeter of the base of a cube $= 12.32^m$; find its surface and its volume.

Ex. 476. Compute the surface and volume of a cube if its diagonal $= 6^m$.

Ex. 477. How many cubes having for an edge 0.9^m can be made from a cube whose edge is 7.2^m?

Ex. 478. The sum of the edges and the diagonal of a cube together is equal to a; find an expression for its volume.

Ex. 479. The edge of a cube is 6^m; find the edge of a cube twice as great.

Ex. 480. Find the diagonal of a cube, the volume being 21.952 cubic metres.

Ex. 481. From 1000^k of iron how many cubes with the edge 0.24^m can be cast? (Specific gravity of the iron, 7.2.)

Ex. 482. A cube of iron weighs 100^k; find its edge. (Specific gravity of iron, 7.2.)

Ex. 483. A cube is cast composed of 11^k of copper and 4.5^k of tin; find its edge. (Specific gravity of copper, 8.8; of tin, 7.3.)

Ex. 484. The side of the regular base of a right triangular pyramid $= a$, a slant edge $= b$; find the entire surface.

Ex. 485. Each of the six edges of a triangular pyramid is 4^m long; find the entire surface.

Ex. 486. The height of a right pyramid is h, the base is a square with the side a; find the entire surface.

Ex. 487. Find the surface of a right pyramid with a square base, the height being 4.8^m, and a side of the base being 7.2^m.

Ex. 488. In a right pyramid, given a side a of the square base, and the entire surface S; find the height.

Ex. 489. The surface of a right quadrangular pyramid with equal edges is 68.3 square metres; find the edges.

Ex. 490. The surface of a right pyramid with square base $= 410.13$ square metres, and a side of the base is to a slant edge as $4:16$; find the side of the base.

Ex. 491. Two homologous sides of the bases of the frustum of a pyramid are a and b $(a > b)$, and the height of the entire pyramid is h; find the height of the frustum.

Ex. 492. Find the entire surface of the frustum of a pyramid with a square base, if a, b, and h denote a side of the greater base, a side of the lesser base, and the height of the frustum, respectively.

Ex. 493. The surface of the frustum of a right pyramid with a square base $= 351.9$ square metres, the height of the frustum $= 10^m$; find a side of each base of the frustum, given that the difference of the two sides is 3^m.

Ex. 494. A side of the square base of a right pyramid measures 2.3^m, the height of the pyramid measures 8.7^m; find the volume.

Ex. 495. Find the volume of a regular hexagonal pyramid 18^m high, a side of the base being 3^m.

Ex. 496. A right pyramid has a square base one side of which $= 3.2^m$, and a slant edge $= 15.8^m$; find the volume.

Ex. 497. Find the volume of a triangular pyramid each edge of which $= 6^m$.

Ex. 498. Find the volume of a right pyramid with a square base, if a side of the base $= a$, and the surface of the pyramid $= S$.

Ex. 499. A right hexagonal pyramid with regular base contains 96 cubic metres, and each slant edge is twice as long as a side of the base; find the slant edge and a side of the base.

Ex. 500. A granite gravestone has the shape of a right pyramid with square base. What is its weight if a side of the base $= 0.9^m$, and a slant edge $= 2.4^m$? (Specific gravity of granite, 2.7.)

Ex. 501. How high must a right pyramid with square base be to weigh 100^k, a side of the base being 0.45^m, and the specific gravity of the material being 2.5?

Ex. 502. In the frustum of a pyramid the areas of the bases are 16 and 9 square metres, and the height of the frustum is 8.7^m; find its volume.

Ex. 503. Find the volume of the frustum of a right pyramid with square base, if a and b are the greater and less sides of the bases, and c one of the slant edges.

Ex. 504. The volume of a frustum of a pyramid = 21.546 cubic metres, the sides of its bases, which are squares, are 2.7^m and 1.8^m; find its height.

Ex. 505. The edge of a regular polyhedron is k; find the radii R and r of the circumscribed and inscribed spheres, the surface S, and the volume V.

The results are as follows: —

Tetrahedron.	*Cube.*	*Octahedron.*
$R=\frac{k}{4}\sqrt{6}.$	$R=\frac{k}{2}\sqrt{3}.$	$R=\frac{k}{2}\sqrt{2}.$
$r=\frac{k}{12}\sqrt{6}.$	$r=\frac{k}{2}.$	$r=\frac{k}{6}\sqrt{6}.$
$S=k^2\sqrt{3}.$	$S=6k^2.$	$S=2k^2\sqrt{3}.$
$V=\frac{k^3}{12}\sqrt{2}.$	$V=k^3$	$V=\frac{k^3}{3}\sqrt{2}.$

Dodecahedron.	*Icosahedron.*
$R=\frac{k}{4}\sqrt{3}(1+\sqrt{5}).$	$R=\frac{k}{4}\sqrt{10+2\sqrt{5}}.$
$r=\frac{k}{4}\sqrt{\frac{5+2\sqrt{5}}{5}}.$	$r=\frac{k}{4}\cdot\frac{3+\sqrt{5}}{\sqrt{3}}.$
$S=3k^2\sqrt{5(5+2\sqrt{5})}.$	$S=5k^2\sqrt{3}.$
$V=\frac{k^3}{4}(15+7\sqrt{5}).$	$V=\frac{5k^3}{12}(3+\sqrt{5}).$

Ex. 506. The edge of a regular tetrahedron = 2.5^m; find its surface and its volume.

Ex. 507. If the height of a regular tetrahedron is 4.75^m, find its surface.

Ex. 508. The difference between the edge and the height of a regular tetrahedron is 1.4^m; find its surface.

Ex. 509. Find the volume of a regular tetrahedron if its height is 5.6^m.

Ex. 510. The sum of the height and the edge of a regular tetrahedron $= 24^m$; find the volume.

Ex. 511. The sum of the volumes of two regular tetrahedrons $= 4.12457$ cubic metres, the sum of two edges $= 5^m$; find the edges.

Ex. 512. In an octahedron whose edge $= k$, a cube is so inscribed that its corners fall in the eight edges of the octahedron; find the volume of this cube.

Ex. 513. Find the surface and volume of a regular octahedron if the edge $= 1.6^m$.

Ex. 514. Find the edge of a regular octahedron if the volume $= 12.728$ cubic metres.

Ex. 515. Find the surface and volume of a regular dodecahedron, the edge being 0.5^m.

Ex. 516. The surface of a regular dodecahedron $= 516$ square metres; find the volume.

Ex. 517. Find the surface and volume of a regular icosahedron if the edge $= 4.6^m$.

Ex. 518. The volume of a regular icosahedron $= 471.24$ cubic metres; find the edge.

Ex. 519. Find the entire surface of a right cylinder, the height being 18^m, and the diameter of the base 2.5^m.

Ex. 520. Height of a cylinder $= 18^m$, perimeter of the base $= 7.85^m$; find the entire surface.

Ex. 521. The convex surface of a right cylinder 3.75^m high contains 7.425 square metres; find the diameter of the cylinder.

Ex. 522. Find the diameter of a cylinder 14^m high when the entire surface contains 119.7125 square metres.

Ex. 523. The sum of the surfaces of two similar cylinders is 24 square metres, their heights are 2.5^m and 1.4^m; find the surface of each.

Ex. 524. Find the volume of a cylinder 12^m high, the radius of the base being 1.5^m.

Ex. 525. Find the volume of a cylinder, the height being 15.4^m, and the perimeter of the base 26.69^m.

Ex. 526. The same exercise, the height being h, and the perimeter of the base being P.

Ex. 527. How many litres will a cylindrical vessel hold 54^{cm} wide and 90^{cm} high?

Ex. 528. If the volume of a cylinder is 942 cubic metres, and the radius of the base 3.75^m, find the height.

Ex. 529. A cylindrical vessel has a diameter of 30^{cm}, and holds 15 litres; how high must it be?

Ex. 530. If the convex surface of a cylinder is M, the height h, find the volume.

Ex. 531. Express the volume of a cylinder of equal thickness and height in terms of S, the entire surface.

Ex. 532. If S denote the entire surface of a cylinder, d the diameter of the base, find the volume.

Ex. 533. Let P denote the perimeter of the base of a cylinder, S the entire surface; find the volume.

Ex. 534. The volume of a cylinder $= V$, the height $= h$; find the entire surface S.

Ex. 535. The convex surface of a cylinder $= M$, the volume $= V$; find the height, and the diameter of the base.

Ex. 536. The entire surface of a right cylinder $= S$, the height $= h$; find the volume.

Ex. 537. An iron weight, weighing 50^{k}, is to be cast; how high will it be if its diameter is made equal to 18^{cm}? (Specific gravity of iron, 7.2.)

Ex. 538. Find the volume of a cylindrical shell 18^{m} high, the exterior diameter being 1.2^{m}, and the width of the bore 0.6^{m}.

Ex. 539. The volume of a cylindrical shell $= V$, the height $= h$, the exterior diameter $= R$; find the diameter of the bore.

Ex. 540. The radius of the base of a right cone $= 34^{cm}$, the slant height $= 1.4^{m}$; find the curved surface, and the entire surface.

Ex. 541. Find the entire surface of a cone in terms of A, the area of the base, and h, the height.

Ex. 542. The height of a right cone $= h$, the slant height $= k$; find the entire surface.

Ex. 543. The convex surface of a cone contains 427.04 square metres, the slant height is 17^m; find the height and the diameter of the base.

Ex. 544. The slant height of a cone is 7.44^m longer than the perimeter of the base; find the diameter of the base, the entire surface being 138.16 square metres.

Ex. 545. The entire surface of a right cone = 75.36 square metres, and the slant height exceeds the diameter of the base by 6^m; find the slant height and the diameter of the base.

Ex. 546. The entire surface of a right cone contains 13.345 square metres. If the slant height and the diameter of the base were each 1^m longer, the surface of the new cone would contain 31.4 square metres; find the slant height and diameter.

Ex. 547. Find the volume of a right cone in which the perimeter of the base = 6.28^m, the slant height = 5.4^m.

Ex. 548. The diameter of the base of a right cone = 6.4^m, the volume = 160.768 cubic metres; how high is it?

Ex. 549. Express the volume of a cone in terms of its entire surface S, and the radius r of the base.

Ex. 550. Express the volume of a right cone in terms of the entire surface S, and the convex surface M.

Ex. 551. A right cone, radius of base = 2.5^m, slant height = 6.5^m, is divided into equal parts by a section parallel to the base; find the height and radius of the base of the upper portion.

Ex. 552. An iron cone weighs 50^k, and the radius of its base $= 12^{cm}$; find its height. (Specific gravity of iron, 7.2.)

Ex. 553. Find the convex surface of the frustum of a cone, the perimeters of the bases being 68.75^m and 185.5^m, and the slant height 23.2^m.

Ex. 554. Find the surface of the frustum of a cone, the height being 2.4^m, the diameters of the bases 6^m and 4^m.

Ex. 555. The entire surface of a frustum of a cone $=$ 96.712 square metres, the diameters of the bases are 6^m and 2^m; find the height of the frustum.

Ex. 556. Find the volume of the frustum of a cone 5.7^m high, if the radii of the bases are 1.4^m and 0.8^m.

Ex. 557. A kettle has for its upper diameter 102^{cm}, and for its lower 84^{cm}, and is 72^{cm} deep; how much water will it hold?

Ex. 558. The perimeters of the bases of the frustum of a right cone are 7.85^m and 6.28^m, the slant height is 5^m; find its volume.

Ex. 559. The volume of a frustum of a cone $= 1186.92$ cubic metres, the height $= 18^m$, the radius of the greater base $= 6^m$; find the radius of the other base.

Ex. 560. The radius of the greater base of a frustum of a cone $= 3^m$, that of the smaller base $= 1^m$, and the height $= 4^m$; if the frustum is divided into equal parts by a plane parallel to the bases, at what distance from the greater base must the section be made, and what will be its radius?

Ex. 561. The trunk of a tree 9^m long has the shape of a frustum of a cone. At one end the perimeter is 4.71^m, at the other 3.768^m; find the weight of the trunk. (Specific gravity of the wood, 0.65.)

Ex. 562. Find the surface of a sphere, the radius being 4.5^m.

Ex. 563. The edge of a regular tetrahedron $= a$; find the surfaces of the circumscribed and inscribed spheres.

Ex. 564. A cube contains 3.375 cubic metres; find the surface of the circumscribed sphere.

Ex. 565. The edge of a regular octahedron $= a$; find the surfaces of the circumscribed and inscribed spheres.

Ex. 566. The same exercise, substituting dodecahedron for octahedron.

Ex. 567. The same exercise, substituting icosahedron for octahedron.

Ex. 568. Find the radius of a sphere which shall have the same extent of surface as two spheres, one with the radius 4.5^m, the other having for the circumference of a great circle 15.7^m?

Ex. 569. The sum of the diameters of two spheres $= d$, the sum of their surfaces $= S$; find the diameter of each.

Ex. 570. The sum of the surfaces of two spheres $=$ 1.413 square metres, the difference of their radii $= 15^{cm}$; find the surface of each.

Ex. 571. How much greater is the volume of a sphere than that of the regular inscribed tetrahedron, if the edge of the latter $= 8^m$?

Ex. 572. The radii of two spheres differ by 1^m, their surfaces differ by 62.8 square metres; find the volume of each sphere.

Ex. 573. If a square and an equilateral triangle are so circumscribed about a circle that one side of the triangle coincides in direction with one side of the square, and the whole is then made to revolve about the altitude of the triangle upon this common side as base, there will be generated by the revolution a sphere, a cylinder, and a cone. Find the ratios of the volumes of these three bodies.

Ex. 574. If two spheres with the diameters 18^m and 36^m are melted and recast into one sphere, find its diameter.

Ex. 575. Find the radius of a sphere which has the same volume as a regular tetrahedron 4^m high.

Ex. 576. The volumes of two spheres are as $m:n$, the diameter of the first $= d$; find that of the other.

Ex. 577. Find the ratio of the volumes of two spheres if that of their surfaces is as $m:n$.

Ex. 578. Find the ratio of the surfaces of two spheres if that of their volumes is as $m:n$.

Ex. 579. The sum of the volumes of two spheres $=$ 1.172 cubic metres, the sum of their radii $= 1^m$; find their diameters.

Ex. 580. The outer surface of a wooden ball $= 6.28$ square metres, the volume of the hollow part $= 904.32$ cubic metres; find the thickness of the wooden shell.

Ex. 581. How many bullets 9^{mm} diameter can be made from half a kilogramme of lead. (Specific gravity of lead, 11.38.)

Ex. 582. At the distance a from the centre of a sphere, a section is made by a plane; find the area of the two spherical segments into which the sphere is divided, if the radius of the circular section is equal to r.

Ex. 583. Find the radius of a sphere if a zone 1.4^{m} high contains 21.98 square metres.

Ex. 584. Find the volume of a spherical sector if the radius of the sphere $= 2.6^{m}$, and the height of the corresponding zone $= 1.6^{m}$.

Ex. 585. What height has the base of a spherical sector, if it contains 0.848 cubic metres, and the radius of the sphere $= 0.9^{m}$?

Ex. 586. The radius of a sphere $= R$, a segment with one base has for this base a circle with the radius r; find the surface and volume of the segment when it is less than a hemisphere.

Ex. 587. Find the volume of a spherical segment if the radii of its bases are 8^{m} and 3^{m}, and its height is 6^{m}.

Ex. 588. A sphere, radius $= 2.4^{m}$, is cut by two parallel planes that make circular sections having the radii 4^{m} and 2^{m}; find the surface and volume of the segment contained between the planes.

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